Technical Report No. 32-718

Tests of Hypotheses and Estimation of the Correlation Coefficient Using Quantiles I

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PASADENA, CALIFORNIA
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PREFACE

An investigation into the use of sample quantiles for data compression of space telemetry was initiated at the Jet Propulsion Laboratory in 1962 at the suggestion of M. Easterling, Manager, Communications Systems Research Section. This work was undertaken by the author in collaboration with Dr. Edward C. Posner, Research Group Supervisor. At that time, our goal was a modest one: we wanted to use a comparatively small number of "sample quantiles" instead of the entire set of sample values taken from a normally distributed population, in order to obtain unbiased estimators of the mean and standard deviation of the parent population; and we wanted to do this as efficiently as possible. Previous work along these lines led us to believe that an investigation of this kind would be fruitful. Starting from scratch and adopting a somewhat different approach to the problem, we obtained results that verified and extended those achieved by previous investigators in the field, notably F. Mosteller and J. Ogawa.

Statistical analysis, however, is not limited to parameter estimation. With a number of sample quantiles at one's disposal, it is reasonable to assume that some additional statistical information can be extracted from them. Lending encouragement to this notion was the fact that, since we were assuming large sample sizes, we were also assuming the asymptotic distribution of the quantiles, which had the useful property of normality. The first results of the investigation into the possible uses of sample quantiles were two "goodness-of-fit" tests, one of which was designed for high power against bimodal distributions. To our knowledge, both tests were entirely new, even as theoretical results.

Although some of our results had been described previously in JPL Space Programs Summary articles, a complete report of the results of our investigation up to that time was presented in the form of a JPL Technical Report (Ref. 1). This Report described the goodness-of-fit tests, the derivation of estimators of the mean using $1, 2, 3, 4, 6, \cdots, 20$ optimal quantiles, and the derivation of estimators of the standard deviation using $2, 4, 6, \cdots, 20$ optimal quantiles under various restrictions. The Report also described, in broad outline, a quantile system designed by Easterling that would select the values of the quantiles from the set of samples, using no arithmetic operations.[1] A refined version of this so-called Quantiler has been designed by Tage Anderson and Warren Lushbaugh (Ref. 2) and has since been built. Posner (Ref. 3) shows that a significant amount of data compression, accompanied by high efficiencies, can be achieved using quantiles, and also suggests several possible applications of the use of quantiles to civilian technology.

By this time we were convinced that the possible statistical uses to which sample quantiles could be applied were far from being exhausted, and it was suggested that the author continue the investigation. This Report presents the first results of this continued investigation.

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[1]A condensed version of this Report has been published in the Journal of the American Statistical Association, Vol. 60, pp 97-133, March 1965.

PREFACE (Cont'd)

It is not uncommon in scientific research that what begins as a more or less routine project may well develop into what perhaps can be best described as a scientific crusade. I find that my work with quantiles almost falls into this category. Objectively, it would be difficult to imagine an entity more prosaic than a sample quantile. It is an effort even to define one properly. Yet, the more one studies these "inefficient" statistics, the more difficult it is to retain one's objectivity. They tend eventually to appear as old friends, steadfast in their dependability, full of pleasant surprises, and having a multitude of uses. The realization that a few of them, properly chosen, can do the work of many helps to foster this illusion. As a consequence of my work with them, my feelings concerning their usefulness and versatility can be best expressed by slightly altering the words of O. Henry's Mrs. Sampson (as immortalized by M. G. Kendall) to read, "I think quantiles are just as lovely as they can be."

<div align="right">I. Eisenberger</div>

Jet Propulsion Laboratory
15 March, 1965

ABSTRACT

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This Report presents some of the results of the continuing investigation into the use of sample quantiles for data compression of space telemetry. The first two tests are concerned with the value of one of the parameters of a normal distribution when the value of the other is known. The third one tests both parameters simultaneously. The next two tests are two sample tests; one tests the hypothesis that $\mu_2 = \mu_1$, and the other tests the hypothesis that $\sigma_2 = \sigma_1$. Finally, pairs of sample values are tested for independence and, in addition, estimators of the correlation coefficient are given. The tests and estimators are all based on sample quantiles, and the sample sizes are assumed to be large.

I. INTRODUCTION

One method of achieving data compression of space telemetry is to transmit a comparatively small number of *sample quantiles* to Earth instead of all the sample values resulting from a space experiment, in order to perform the identical statistical analysis for which the set of sample values was originally intended. It is obvious that the transmission of, say, four quantiles instead of 1000 sample values will result in a large data compression ratio. But very little can be gained by using this method unless:

1. The loss of information (as defined according to reasonable criteria) is small,

2. The mechanization of a quantile system aboard the spacecraft to select and transmit the required sample quantile values is simpler than the data processing equipment necessary to perform the final statistical analyses in the same environment and transmit the results to Earth.

It has been shown that the above two conditions necessary to the effective use of sample quantiles for data compression can be more than satisfactorily met (Ref. 1, 2, and 3). It is evident, however, that the advantage gained by using quantiles would be augmented if further uses for them were developed. Assuming normally distributed populations throughout, several tests of hypotheses are given, as well as estimators of the correlation between two populations. First, we assume that the standard deviation σ is known and test whether the mean μ has a value μ_1 or a value μ_2. Next, we assume that μ is known and test whether $\sigma = \sigma_1$ or $\sigma = \sigma_2$. The third test determines whether $\mu = \mu_1$ *and* $\sigma = \sigma_1$ or whether $\mu = \mu_2$ *and* $\sigma = \sigma_2$. For the next two tests, we assume that we are given sets of independent samples taken from two independent normal populations, and we consider the following problems:

1. If $\sigma = \sigma_1 = \sigma_2$ is known and μ_1 is not known, is $\mu_2 = \mu_1$ or is $\mu_2 = \mu_1 + \theta$, $\theta \neq 0$?

2. If μ_1 and μ_2 are known and σ_1 is not known, is $\sigma_2 = \sigma_1$ or is $\sigma_2 = \theta\sigma_1$, $\theta > 0$?

Finally, we assume n independent pairs of observations taken from two normal populations with known means and variances, and we test for independence. In addition, estimators of the correlation coefficient ρ are given. In all cases, the tests and estimators are based on sample quantiles, and the sample sizes are assumed to be large.

Since it is important to know how "good" the quantile tests are compared to the best tests using all the sample values, the power functions P_0 of the quantile tests and the power functions P_0' of the best tests using all the sample values are derived. The efficiencies of the quantile tests, P_0/P_0', are then determined. The efficiencies $\mathrm{Var}\,(\hat{\rho})/\mathrm{Var}\,(r)$ of $\hat{\rho}$, which are the estimators of ρ using quantiles, are also determined for the special case $\rho = 0$, where r is the sample correlation coefficient.

The uncertainty that invariably accompanies the application of statistical techniques to problems of estimation and prediction is usually an inverse function of the sample size, n. One of the common measures of this uncertainty is the variance of the particular statistic used. Most of the useful standard statistics have the property that their variances decrease with an increase in the sample size, with a consequent reduction in the uncertainty inherent in the conclusions based on them. Because the variances of sample quantiles are, asymptotically, inversely proportional to n, the same reduction in uncertainty, when n is increased, follows from the use of these order statistics as from the use of non-ordered ones. Thus, the principal advantage of a large sample size is not sacrificed by this form of data compression.

II. REVIEW OF QUANTILES

To define a quantile, consider a sample of n independent values $x_1, x_2, \cdots, x_n$ taken from a distribution of a continuous type with distribution function $G(x)$ and density function $g(x)$. The pth quantile, or the quantile of order p of the distribution or population, denoted by ζ_p, is defined as the root of the equation $G(\zeta) = p$; that is,

$$p = \int_{-\infty}^{\zeta_p} dG(x) = \int_{-\infty}^{\zeta_p} g(x)\,dx$$

The corresponding *sample* quantile, z_p, is defined as follows:

If the sample values are arranged in non-decreasing order of magnitude

$$x_{(1)} \leqslant x_{(2)} \leqslant \ldots \leqslant x_{(n)}$$

then $x_{(i)}$ is called the ith order statistic and

$$z_p = x_{[np]+1}$$

where $[np]$ is the greatest integer $\leqq np$.

If $g(x)$ is differentiable in some neighborhood of each quantile value considered, it has been shown (Ref. 4)

that the joint distribution of any number of quantiles is asymptotically normal as $n \to \infty$ and that, asymptotically,

$$E(z_p) = \zeta_p$$

$$\mathrm{Var}\,(z_p) = \frac{p(1-p)}{ng^2(\zeta_p)}$$

$$\rho_{12} = \left[\frac{p_1(1-p_2)}{p_2(1-p_1)}\right]^{\frac{1}{2}}$$

where ρ_{12} is the correlation between z_{p_1} and z_{p_2}, $p_1 < p_2$.

Throughout this Report we will denote by $F(x)$ and $f(x) = F'(x)$ the distribution function and density function, respectively, of the standard normal distribution; that is,

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

where

$$f(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-x^2/2}$$

The statement "$g(x) = N(\mu, \sigma)$" will mean that the random variable under consideration is normally distributed with mean μ, variance σ^2, and has the density function $g(x)$ associated with it. For simplification, when only

one quantile is being considered, the sample quantile of order p will be denoted by z, the corresponding population quantile by ζ^*, and the corresponding population quantile of the standard normal distribution by ζ. Thus, one has

$$p = \int_{-\infty}^{\zeta^*} g(x)\, dx = \frac{1}{\sigma (2\pi)^{1/2}} \int_{-\infty}^{\zeta^*} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}\, dx$$

$$= \int_{-\infty}^{\zeta^*-\mu/\sigma} f(x)\, dx = \int_{-\infty}^{\zeta} f(x)\, dx$$

Hence, one sees that, asymptotically,

$$E(z) = \zeta^* = \sigma\zeta + \mu$$

and, since $g(\zeta^*) = 1/\sigma f(\zeta)$,

$$\text{Var}(z) = \frac{\sigma^2 F(\zeta)\,[1 - F(\zeta)]}{nf^2(\zeta)}$$

so that the moments of the sample quantiles of normal distributions are expressible in terms of the standard normal distribution. In this Report, when m quantiles are being considered, the sample quantiles will be denoted by z_i of order p_i ($i = 1, 2, \cdots, m$) and $p_i < p_j$ for $i < j$. ζ_i will denote the corresponding population quantile of the standard normal distribution. The tests of hypotheses to be given will be denoted by Test A, Test B, Test C, etc. Test A_i will denote Test A using i quantiles, Test B_i will denote Test B using i quantiles, and so on.

III. TEST A: TESTING THE MEAN OF A NORMAL DISTRIBUTION USING QUANTILES

A. Test A_1: One Quantile

Suppose one is given a set of n sample values

$$x_1, x_2, \cdots, x_n,$$

taken from a normally distributed population with density function $g(x)$, and one wishes to test the simple null hypothesis

$$H_0: \quad g(x) = g_1(x) = N(\mu_1, \sigma)$$

against the simple alternative hypothesis

$$H_1: \quad g(x) = g_2(x) = N(\mu_2, \sigma)$$

where $\mu_2 > \mu_1$ ($\mu_2 < \mu_1$) and σ is known. If $\mu_1 = 0$, this test would correspond to the problem of detecting a dc signal of known amplitude in the presence of additive stationary Gaussian noise. Although it is well known that the optimum test is based on the sum of the observations, the tests given here will be based on the values of one, two, and four sample quantiles.

Beginning with one quantile, let z denote the sample quantile of order p, and let ζ be the corresponding population quantile of the standard normal. Then one has,

Under H_0:

$$E(z) = \sigma\zeta + \mu_1$$

$$\text{Var}(z) = \sigma^2 a^2$$

where

$$a^2 = \frac{F(\zeta)\,[1 - F(\zeta)]}{nf^2(\zeta)}$$

Under H_1:

$$E(z) = \sigma\zeta + \mu_2$$

$$\text{Var}(z) = \sigma^2 a^2$$

The best critical (or rejection) region is determined by the *likelihood ratio* inequality (Ref. 5, p. 166) which, assuming the limiting distribution of z, is given by

$$\frac{L(z|H_0)}{L(z|H_1)} = \frac{\exp\left[-\dfrac{1}{2}\left(\dfrac{z - \sigma\zeta - \mu_1}{\sigma a}\right)^2\right]}{\exp\left[-\dfrac{1}{2}\left(\dfrac{z - \sigma\zeta - \mu_2}{\sigma a}\right)^2\right]} < c$$

By taking logarithms and simplifying, this inequality reduces to

$$z > k \qquad (\mu_2 > \mu_1)$$

as the region providing the maximum power. Here k is determined such that, under H_0, $\Pr(z > k) = \varepsilon$, the significance level of the test; that is, ε is the probability of rejecting H_0 when H_0 is true.

The power of a test of this type is defined to be the probability of rejecting H_0 when H_1 is true, and thus depends critically upon the nature of H_1 as well as upon ε. Since the efficiency has been defined as P_0/P_0', the ratio of the power of the test using quantiles to the power of the best test using the entire sample, the power functions P_0 and P_0' will be determined.

Under H_0:

$$\Pr(z < k) = F\left(\frac{k - \sigma\zeta - \mu_1}{\sigma a}\right) = F(b) = 1 - \varepsilon$$

$$k = \sigma ab + \sigma\zeta + \mu_1$$

Under H_1:

$$\Pr(z < k) = F\left(\frac{k - \sigma\zeta - \mu_2}{\sigma a}\right)$$

$$= F\left(\frac{\sigma ab + \sigma\zeta + \mu_1 - \sigma\zeta - \mu_2}{\sigma a}\right)$$

$$= F\left(b - \frac{\mu_2 - \mu_1}{\sigma a}\right) = 1 - P_0$$

Since for fixed $\mu_2 - \mu_1/\sigma$, P_0 depends only upon the quantity a, which in turn depends only upon the order of the quantile chosen for the test, it is natural to choose that quantile which minimizes a and hence maximizes P_0. This turns out, not surprisingly, to be the *median*, the quantile of order .5. For this value of p, $a = 1.253/n^{1/2}$, and thus the optimum power function is given by

$$P_0 = 1 - F\left(b - .798\, n^{1/2}\frac{\mu_2 - \mu_1}{\sigma}\right)$$

It is easily shown that P_0' is given by

$$P_0' = 1 - F\left(b - n^{1/2}\frac{\mu_2 - \mu_1}{\sigma}\right)$$

Figure 1, for $n = 200$ and $\varepsilon = .01$, shows the power and efficiency of Test A_1 using one optimum quantile. Under these conditions,

$$b = 2.326 \quad \text{and} \quad k = \mu_1 + 2.915\sigma/n^{1/2} = \mu_1 + .2061\sigma.$$
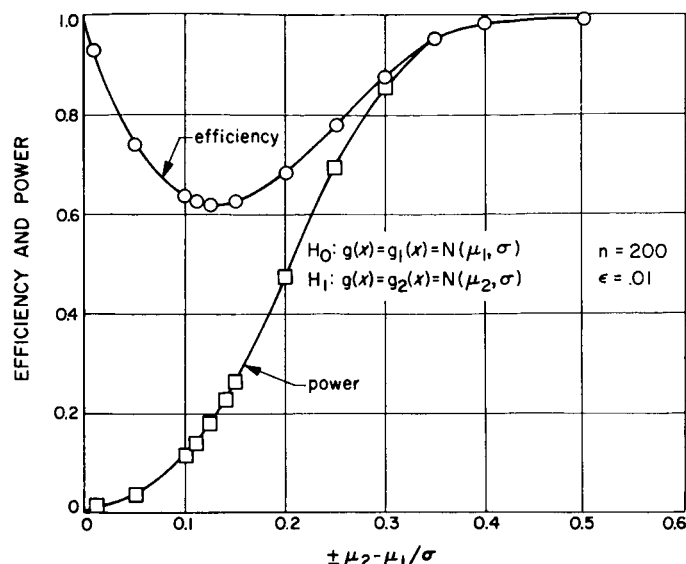
**Fig. 1. Power and efficiency of Test A_1
using one quantile**

Thus, for $\mu_2 > \mu_1$, if the median of the sample does not exceed $\mu_1 + 2.915\sigma/n^{1/2}$, accept H_0. Otherwise reject H_0. The test is made at a significant level of .01.

B. Test A_2: Two Quantiles

Let z_1 and z_2 denote the sample quantiles of orders p_1 and $p_2 = 1 - p_1$. Then one has

Under H_0:

$$E(z_1) = -\sigma\zeta_2 + \mu_1$$

$$E(z_2) = \sigma\zeta_2 + \mu_1$$

$$\text{Var}(z_1) = \text{Var}(z_2) = \sigma^2 a^2$$

where

$$a^2 = \frac{F(\zeta_2)[1 - F(\zeta_2)]}{nf^2(\zeta_2)}$$

Under H_1:

$$E(z_1) = -\sigma\zeta_2 + \mu_2$$

$$E(z_2) = \sigma\zeta_2 + \mu_2$$

$$\text{Var}(z_1) = \text{Var}(z_2) = \sigma^2 a^2$$

For this case, the likelihood function $L(z_1, z_2 | H_0)$ is the joint distribution of z_1 and z_2, given that H_0 is true; similarly, $L(z_1, z_2 | H_1)$ is the joint distribution of z_1 and

z_2, given that H_1 is true. Simplifying the likelihood ratio inequality,

$$\frac{L(z_1, z_2|H_0)}{L(z_1, z_2|H_1)} = \frac{\exp\left\{-\frac{1}{2}\left[\frac{(z_1 + \sigma\zeta_2 - \mu_1)^2 - 2\rho(z_1 + \sigma\zeta_2 - \mu_1)(z_2 - \sigma\zeta_2 - \mu_1) + (z_2 - \sigma\zeta_2 - \mu_1)^2}{\sigma^2 a^2}\right]\right\}}{\exp\left\{-\frac{1}{2}\left[\frac{(z_1 + \sigma\zeta_2 - \mu_2)^2 - 2\rho(z_1 + \sigma\zeta_2 - \mu_2)(z_2 - \sigma\zeta_2 - \mu_2) + (z_2 - \sigma\zeta_2 - \mu_2)^2}{\sigma^2 a^2}\right]\right\}} < c$$

where ρ denotes the correlation between z_1 and z_2, results in the best critical region

$$y = z_1 + z_2 > k \qquad (\mu_2 > \mu_1)$$

Under H_0:

$$E(y) = 2\mu_1$$

$$\text{Var}(y) = \sigma_y^2 = 2\sigma^2 a^2 (1 + \rho)$$

$$\text{Pr}(y < k) = F\left(\frac{k - 2\mu_1}{\sigma_y}\right) = F(b) = 1 - \varepsilon$$

where

$$k = \sigma_y b + 2\mu_1$$

Under H_1:

$$E(y) = 2\mu_2$$

$$\sigma_y^2 = 2\sigma^2 a^2 (1 + \rho)$$

$$\text{Pr}(y < k) = F\left(\frac{k - 2\mu_2}{\sigma_y}\right) = F\left[b - \frac{2(\mu_2 - \mu_1)}{\sigma_y}\right] = 1 - P_0$$

Minimizing

$$\sigma_y^2 = \frac{2\sigma^2 [1 - F(\zeta_2)]}{n f^2 (\zeta_2)}$$

by setting equal to zero the derivative of this quantity with respect to ζ, one obtains the orders of the two symmetric quantiles which maximize P_0. These are

$$p_1 = .2703 \qquad p_2 = .7297$$

so that the optimum power function is

$$P_0 = 1 - F\left(b - .90 \, n^{1/2} \frac{\mu_2 - \mu_1}{\sigma}\right)$$

We are only considering symmetric quantiles; they have been shown (Ref. 6, Table 10E.1) to have the optimum spacing for estimating the mean of a normal population by using an even number of quantiles.
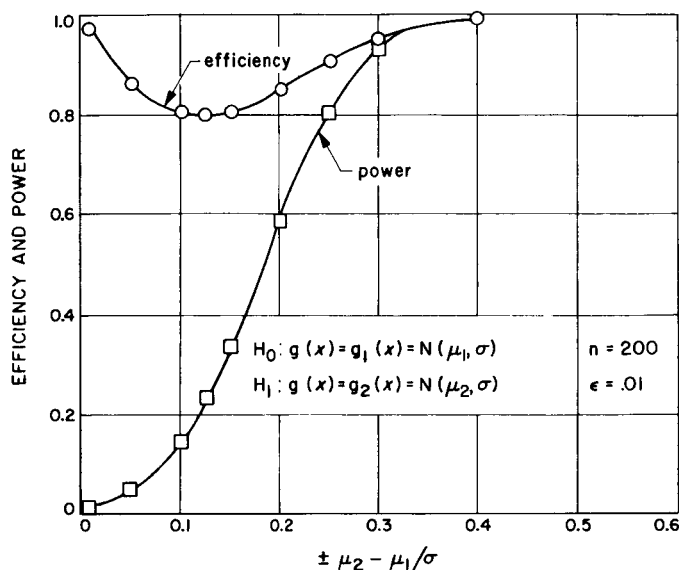
Fig. 2. Power and efficiency of Test A_2 using two quantiles

For $n = 200$, $\varepsilon = .01$, Fig. 2 shows the power and efficiency of Test A_2 using the two optimal quantiles. For this case,

$$k = 2\mu_1 + \frac{5.1693\sigma}{n^{1/2}} = 2\mu_1 + .3655\sigma$$

Hence, for $\mu_2 > \mu_1$, if

$$y = z(.2703) + z(.9297) < 2\mu_1 + \frac{5.1693\sigma}{n^{1/2}}$$

or if

$$y' = \frac{1}{2}\left[z(.2703) + z(.9297) \right] < \mu_1 + \frac{2.585\sigma}{n^{1/2}}$$

accept H_0. Otherwise reject H_0.

C. Test A_i: Four Quantiles

The four-quantile case is slightly more complicated but still straightforward. Let z_i ($i = 1, 2, 3, 4$) denote four sample quantiles such that $p_1 + p_4 = p_2 + p_3 = 1$, and let $x_1 = z_1 + z_4$ and $x_2 = z_2 + z_3$.

Under H_0:

$$E(x_1) = E(x_2) = 2\mu_1$$

$$Var(x_1) = \sigma^2 a_1^2$$

$$Var(x_2) = \sigma^2 a_2^2$$

where

$$a_i^2 = \frac{2F(\zeta_i)}{nf^2(\zeta_i)}$$

$$i = 1, 2$$

Under H_1:

$$E(x_1) = E(x_2) = 2\mu_2$$

$$Var(x_1) = \sigma^2 a_1^2$$

$$Var(x_2) = \sigma^2 a_2^2$$

Omitting the details, the critical region is found to be

$$y = \alpha x_1 + \beta x_2 > k \qquad (\mu_2 > \mu_1)$$

where

$$\alpha = \frac{1}{a_1^2} - \frac{\rho}{a_1 a_2}$$

$$\beta = \frac{1}{a_2^2} - \frac{\rho}{a_1 a_2}$$

and

$$\rho = \left[\frac{F(\zeta_1)}{F(\zeta_2)}\right]^{1/2}$$

denotes the correlation between x_1 and x_2.

Under H_0, for $\mu_2 > \mu_1$:

$$E(y) = 2(\alpha + \beta)\mu_1$$

$$Var(y) = \sigma^2 (\alpha^2 a_1^2 + \beta^2 a_2^2 + 2\rho\alpha\beta a_1 a_2) = \sigma^2 \gamma^2$$

$$Pr(y < k) = F\left[\frac{k - 2(\alpha + \beta)\mu_1}{\sigma\gamma}\right] = F(b) = 1 - \varepsilon$$

$$k = \sigma\gamma b + 2(\alpha + \beta)\mu_1$$

Under H_1:

$$E(y) = 2(\alpha + \beta)\mu_2$$

$$Var(y) = \sigma^2 \gamma^2$$

$$Pr(y < k) = F\left[b - \frac{2(\alpha + \beta)(\mu_2 - \mu_1)}{\sigma\gamma}\right] = 1 - P_0$$

The two pairs of symmetric quantiles which maximize P_0 are those of orders

$$p_1 = .1068 \qquad p_2 = .3512$$

$$p_4 = .8932 \qquad p_3 = .6488$$

The optimum power function using four quantiles is thus given by

$$P_0 = 1 - F\left(b - .959\,n^{\frac{1}{2}}\frac{\mu_2 - \mu_1}{\sigma}\right)$$

For $n = 200$, $\varepsilon = .01$, Fig. 3 shows the power and efficiency of Test A_4 using four optimal quantiles. For this case,

$$\alpha = 12.290$$

$$\beta = 19.732$$

$$k = 64.043\mu_1 + \frac{155.85}{n^{\frac{1}{2}}}$$

so that if

$$y = 12.290\,[z(.1068) + z(.8932)]$$

$$+ 19.732\,[z(.3512) + z(.6488)] < 64.043\mu_1 + \frac{155.85\sigma}{n^{\frac{1}{2}}}$$

or if

$$y' = .192\,[z(.1068) + z(.8932)]$$

$$+ .308\,[z(.3512) + z(.6488)] < \mu_1 + \frac{2.4335\sigma}{n^{\frac{1}{2}}}$$

accept H_0. Otherwise reject H_0.

In each case, since the critical region depends only upon the parameters of $g_1(x)$, the hypothesized population distribution density under H_0, the tests are seen to be *uniformly most powerful* (among quantile tests). If $\mu_1 > \mu_2$ is specified instead of $\mu_2 > \mu_1$, a similar analysis gives the following *acceptance* regions:

$$z(.5) > \mu_1 - \frac{2.915\sigma}{n}$$

$$\frac{1}{2}\,[z(.2703) + z(.7297)] > \mu_1 - \frac{2.585\sigma}{n^{\frac{1}{2}}} \qquad (1)$$

$$.192\,[z(.1068) + z(.8932)]$$

$$+ .308\,[z(.3512) + z(.6488)] > \mu_1 - \frac{2.433\sigma}{n^{\frac{1}{2}}}$$
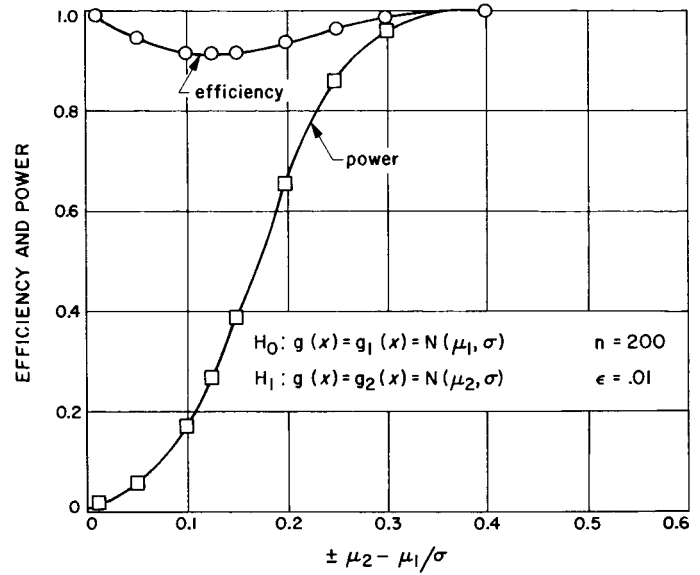
Fig. 3. Power and efficiency of Test A_4 using four quantiles

and if $\mu_2 - \mu_1 = \mu_1 - \mu_3 > 0$, then

$$P_0\left(\frac{\mu_2 - \mu_1}{\sigma}\right) = P_0\left(\frac{\mu_1 - \mu_3}{\sigma}\right)$$

It is interesting to note that the left sides of all the above inequalities (1) are not only unbiased estimators of the true mean μ using one, two, and four quantiles, but they are also the maximum-likelihood estimators of μ with minimum variance when σ is *unknown*, since one observes that the orders of the quantiles which provide the maximum power for each test are also those which minimize the variance of the estimates of μ (Ref. 1, pp. 4–7). This coincidence can also be proved non-computationally.

In summary, the minimum efficiency for the quantile test for testing the mean of a normal distribution with known variance is about .61 for one quantile, about .80 for two quantiles, and about .91 for four quantiles, if the quantiles are chosen optimally. This surprising result and those that follow lend further strength to the feasibility of quantiles for data compression.

IV· TEST B: TESTING THE STANDARD DEVIATION OF A NORMAL DISTRIBUTION USING QUANTILES

A. Test B_1: One Quantile

In Test A we assumed that σ is known, and although μ is unknown, we had some reason to believe that μ is either equal to μ_1 or equal to some other value

$$\mu_2 > \mu_1 \, (\mu_2 < \mu_1)$$

Since both hypotheses assumed the same variance, the test statistics turned out to be linear functions of the quantiles. As a result, the tests were all one-sided. In the present test we are assuming that μ is known, and we wish to discriminate between $\sigma = \sigma_1$ and $\sigma = \sigma_2 > \sigma_1 \, (\sigma_2 < \sigma_1)$. It will be seen that the best tests using quantiles are not one-sided but can be closely approximated by one-sided tests, with a negligible loss in power in most cases.

More precisely, the simple null hypothesis

$$H_0: \quad g(x) = g_1(x) = N(\mu, \sigma_1)$$

against the simple alternative hypothesis

$$H_1: \quad g(x) = g_2(x) = N(\mu, \sigma_2)$$

is considered, where μ is known and $\sigma_2 > \sigma_1 \, (\sigma_2 < \sigma_1)$. The test will be based on one, two, and four quantiles, and in each case the power function is derived and the efficiency determined.

Let z denote the sample quantile of order p. Then

Under H_0:

$$E(z) = \sigma_1 \zeta + \mu$$

$$\mathrm{Var}(z) = \sigma_1^2 a^2$$

Under H_0:

where

$$a^2 = \frac{F(\zeta)\,[1 - F(\zeta)]}{nf^2(\zeta)}$$

Under H_1:

$$E(z) = \sigma_2 \zeta + \mu$$

$$\mathrm{Var}(z) = \sigma_2^2 a^2$$

If the test is now based on the value of z, the best critical region is that for which

$$\frac{L(z|H_0)}{L(z|H_1)} = \frac{\exp\left[-\frac{1}{2}\left(\frac{z - \sigma_1 \zeta - \mu}{\sigma_1 a} \right)^2 \right]}{\exp\left[-\frac{1}{2}\left(\frac{z - \sigma_2 \zeta - \mu}{\sigma_2 a} \right)^2 \right]} < c_1 \tag{2}$$

By taking logarithms, Inequality 2 becomes

$$\frac{(z - \sigma_2 \zeta - \mu)^2}{\sigma_2^2} - \frac{(z - \sigma_1 \zeta - \mu)^2}{\sigma_1^2} < c_2$$

or

$$(\sigma_1^2 - \sigma_2^2) z^2 - 2z\,[\sigma_1^2(\sigma_2^2 \zeta + \mu) - \sigma_2^2(\sigma_1 \zeta + \mu)] < c_3$$

which, by completing the square in z, reduces to

$$\left[z - \left(\mu + \frac{\sigma_1 \sigma_2 \zeta}{\sigma_1 + \sigma_2} \right) \right]^2 > k^2 \qquad (\sigma_2 > \sigma_1) \tag{3}$$

where k^2 is determined such that, under H_0, the probability of Inequality 3 occurring is equal to ε. The power function P_0 is determined as follows:

$$\Pr\left(-k + \mu + \frac{\sigma_1 \sigma_2 \zeta}{\sigma_1 + \sigma_2} < z < k + \mu + \frac{\sigma_1 \sigma_2 \zeta}{\sigma_1 + \sigma_2} \right)$$

$$= F\left(\frac{k + \mu + \frac{\sigma_1 \sigma_2 \zeta}{\sigma_1 + \sigma_2} - \sigma_1 \zeta - \mu}{\sigma_1 a} \right) - F\left(\frac{-k + \mu + \frac{\sigma_1 \sigma_2 \zeta}{\sigma_1 + \sigma_2} - \sigma_1 \zeta - \mu}{\sigma_1 a} \right)$$

$$= F\left(\frac{k + \frac{\sigma_1 \sigma_2 \zeta}{\sigma_1 + \sigma_2} - \sigma_1 \zeta}{\sigma_1 a} \right) - F\left(\frac{-k + \frac{\sigma_1 \sigma_2 \zeta}{\sigma_1 + \sigma_2} - \sigma_1 \zeta}{\sigma_1 a} \right) = F(b) - F(c) = 1 - \varepsilon$$

$$k = \sigma_1 a b + \sigma_1 \zeta - \frac{\sigma_1 \sigma_2 \zeta}{\sigma_1 + \sigma_2} = -\sigma_1 a c - \sigma_1 \zeta + \frac{\sigma_1 \sigma_2 \zeta}{\sigma_1 + \sigma_2} \tag{4}$$

Using the value of k from Eq. 4, the best acceptance region is thus given by

$$\mu - \sigma_1 ab + \frac{\sigma_1 \zeta \left(\frac{\sigma_2}{\sigma_1} - 1 \right)}{\frac{\sigma_2}{\sigma_1} + 1} < z < \mu + \sigma_1 ab + \sigma_1 \zeta$$

Under H_1:

$$\Pr \left(-k + \mu + \frac{\sigma_1 \sigma_2 \zeta}{\sigma_1 + \sigma_2} < z < k + \mu + \frac{\sigma_1 \sigma_2 \zeta}{\sigma_1 + \sigma_2} \right)$$

$$= F \left[\frac{\sigma_1}{\sigma_2} b + \frac{\zeta (\sigma_1 - \sigma_2)}{\sigma_2 a} \right] - F \left[\frac{\sigma_1}{\sigma_2} c + \frac{\zeta (\sigma_1 - \sigma_2)}{\sigma_2 a} \right]$$

$$= F \left[\frac{\sigma_1}{\sigma_2} b - \frac{\zeta}{a} \left(1 - \frac{\sigma_1}{\sigma_2} \right) \right] - F \left[\frac{\sigma_1}{\sigma_2} c - \frac{\zeta}{a} \left(1 - \frac{\sigma_1}{\sigma_2} \right) \right] = 1 - P_0 \qquad (5)$$

Theoretically, the values of b and c depend upon the parameters of both $g_1 (x)$ and $g_2 (x)$, as well as upon ε. However, a few simple calculations show that, assuming the usual small values of ε, one can determine b by the relation $F (b) = 1 - \varepsilon$ and neglect as negligible the second term of the left-hand side of Eq. 5. With this simplification, P_0 will be maximized if the order of the quantile is chosen to maximize ζ/a. Setting equal to zero the derivative of this quantity with respect to ζ, one finds that this maximum occurs at $p = F (1.575) = .9424$. For this value of p, $a = 2.0193/n^{1/2}$ and $\zeta/a = .78 n^{1/2}$. Inserting these values, one obtains

$$P_0 = 1 - F \left[\frac{\sigma_1}{\sigma_2} b - .78 n^{1/2} \left(1 - \frac{\sigma_1}{\sigma_2} \right) \right] + F \left[\frac{\sigma_1}{\sigma_2} c - .78 n^{1/2} \left(1 - \frac{\sigma_1}{\sigma_2} \right) \right] \qquad (6)$$

Noting from Eq. 4 that

$$c = -b - \frac{2\zeta}{a} \left(\frac{1}{1 + \frac{\sigma_2}{\sigma_1}} \right)$$

then under these conditions one has, for the argument of the last term of the right-hand side of Eq. 6,

$$\frac{\sigma_1}{\sigma_2} c - .78 n^{1/2} \left(1 - \frac{\sigma_1}{\sigma_2} \right) = -\frac{\sigma_1}{\sigma_2} b - 1.56 n^{1/2} \frac{\sigma_1}{\sigma_2} \left(\frac{1}{1 + \frac{\sigma_2}{\sigma_1}} \right) - .78 n^{1/2} \left(1 - \frac{\sigma_1}{\sigma_2} \right) < -\frac{\sigma_1}{\sigma_2} b - .646 n^{1/2} < -.646 n^{1/2}$$

(since $b > 0$ for small values of ε), a result which is obtained by maximizing

$$-1.56\, n^{\frac{1}{2}} \frac{\sigma_1}{\sigma_2} \left(\frac{1}{1 + \frac{\sigma_2}{\sigma_1}} \right) - .78\, n^{\frac{1}{2}} \left(1 - \frac{\sigma_1}{\sigma_2} \right)$$

over all values of $\sigma_2/\sigma_1 > 1$. If $n \geqq 200$, then

$$F\left[\frac{\sigma_1}{\sigma_2} c - .78\, n^{\frac{1}{2}} \left(1 - \frac{\sigma_1}{\sigma_2} \right) \right] \leqq F(-9.136) \cong 0$$

and hence contributes nothing to the power of the test, verifying the negligibility of this term. By determining b by the relation $F(b) = 1 - \varepsilon$, the test becomes the best one-sided test and has the advantage of being independent of the parameters of $g_2(x)$. Thus, for $\varepsilon = .01$, if

$$z(.9424) < \mu + \sigma_1 \left(\frac{4.697}{n^{\frac{1}{2}}} + 1.575 \right)$$

accept H_0. Otherwise reject H_0.

To determine P_0', the likelihood ratio inequality in the case of the entire sample gives as the best critical region

$$\sum_{i=1}^{n} (x_i - \mu)^2 > k' \qquad (\sigma_2 > \sigma_1)$$

It is more convenient to express this as

$$y = \left[2 \sum_{i=1}^{n} (x_i - \mu)^2 \right]^{\frac{1}{2}} > k$$

since it is shown (Ref. 4) that, asymptotically, y is distributed $N(\sigma_1(2n)^{\frac{1}{2}}, \sigma_1)$ under H_0 and distributed $N(\sigma_2(2n)^{\frac{1}{2}}, \sigma_2)$ under H_1. Hence, assuming these limiting distributions, which are very good approximations for large n, one has for P_0'

$$P_0' = 1 - F\left[\frac{\sigma_1}{\sigma_2} b - (2n)^{\frac{1}{2}} \left(1 - \frac{\sigma_1}{\sigma_2} \right) \right]$$

Figure 4 shows the efficiency and power of the modified Test B for $n = 200$, $\varepsilon = .01$. The efficiency is never less than .35 and, as in Test A, the efficiency approaches 1 as $\sigma_2/\sigma_1 \to 0$ or ∞.

It is also seen from Fig. 4 that for $\sigma_2/\sigma_1 \geqq 1.5$, the power of the modified Test B_1 is very close to 1 and hence can differ from the power of the two-sided test by very little. Moreover, for $1 < \sigma_2/\sigma_1 < 1.5$, since $E(z|H_0) = \mu_1 + \sigma_1 \zeta$
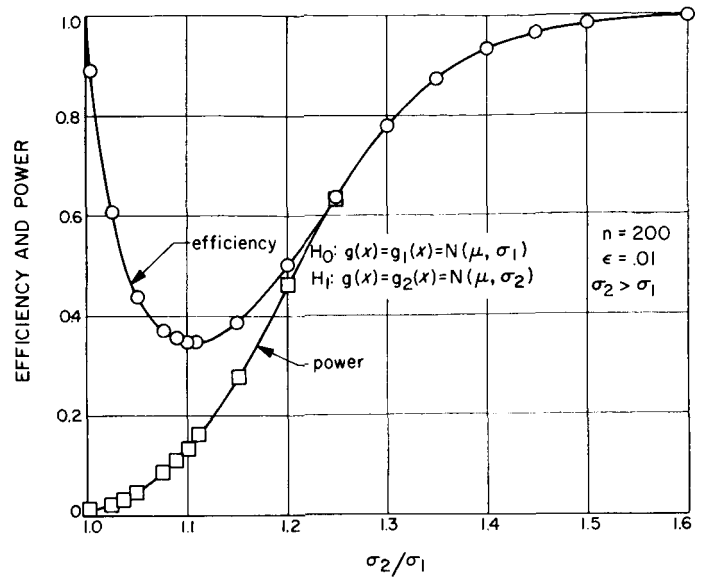
Fig. 4. Power and efficiency of Test B_1 using one quantile

and $\mathrm{Var}(z|H_0) = \sigma_1^2 a^2$, the probability of $z(.9424)$ being less than

$$\mu_1 - \sigma_1 \left[\frac{4.697}{n^{\frac{1}{2}}} - \frac{1.575 \left(\frac{\sigma_2}{\sigma_1} - 1 \right)}{\frac{\sigma_2}{\sigma_1} + 1} \right]$$

the lower bound of the acceptance region of the two-sided test, is practically zero; so that, for these values of σ_2/σ_1,

$$\mathrm{Pr}\left\{ \mu_1 - \sigma_1 \left[\frac{4.697}{n} - 1.575 \left(\frac{\frac{\sigma_2}{\sigma_1} - 1}{\frac{\sigma_2}{\sigma_1} + 1} \right) \right] \right.$$

$$< z(.9424) < \mu_1 + \sigma_1 \left(\frac{4.697}{n} + 1.575 \right) \left. \right\}$$

$$\cong \mathrm{Pr}\left[z(.9424) < \mu_1 + \sigma_1 \left(\frac{4.697}{n} + 1.575 \right) \right]$$

Thus for all practical purposes both tests give almost identical results; hence the one-sided best test is a good substitute for the two-sided best test.

Since $E(z)$ under both hypotheses is a linear function of ζ, an equivalent best one-sided acceptance region can be obtained by using the sample quantile of order

$$p = F(-1.575) = 1 - .9424 = .0576$$

Now we determine c from $F(c) = \varepsilon$. As a result, the first term of the left-hand side of Eq. 5 is practically equal to 1, so that

$$F\left[\frac{\sigma_1}{\sigma_2}c - \frac{\zeta}{a}\left(1 - \frac{\sigma_1}{\sigma_2}\right)\right] \cong P_0$$

and one sees that the power function for this choice of p is identical with that using $\zeta = 1.575$. In effect we have switched the rejection region from the right tail of the density function of z (.9424) (under H_0) to the left tail of the density function of z (.0576). Thus, for $\varepsilon = .01$, if

$$z(.0576) > \mu - \sigma_1\left(\frac{4.697}{n^{\frac{1}{2}}} + 1.575\right)$$

accept H_0. Otherwise reject H_0.

B. Test B_2: Two Quantiles

Now let z_1 and z_2 be two sample quantiles of orders p_1 and $p_2 = 1 - p_1$, and let $x = z_2 - z_1$. The test will be based on the value of the statistic x.

Under H_0:

$$E(x) = 2\sigma_1\zeta_2$$

$$Var(x) = 2Var(z_2)(1 - \rho_{12}) = \frac{2\sigma_1^2 F(\zeta_2)[1 - F(\zeta_2)]}{nf^2(\zeta_2)}\left[1 - \frac{1 - F(\zeta_2)}{F(\zeta_2)}\right] = \frac{2\sigma_1^2[1 - F(\zeta_2)][2F(\zeta_2) - 1]}{nf^2(\zeta_2)} = 2\sigma_1^2 d^2$$

Under H_1:

$$E(x) = 2\sigma_2\zeta_2$$

$$Var(x) = 2\sigma_2^2 d^2$$

Simplifying the likelihood ratio inequality results in the best critical region

$$\left(x - \frac{2\sigma_1\sigma_2\zeta_2}{\sigma_1 + \sigma_2}\right)^2 > k^2 \qquad (\sigma_2 > \sigma_1)$$

To determine P_0, one finds that

Under H_0:

$$Pr\left(-k + \frac{2\sigma_1\sigma_2\zeta_2}{\sigma_1 + \sigma_2} < x < k + \frac{2\sigma_1\sigma_2\zeta_2}{\sigma_1 + \sigma_2}\right) = F\left(\frac{k + \frac{2\sigma_1\sigma_2\zeta_2}{\sigma_1 + \sigma_2} - 2\sigma_1\zeta_2}{2^{\frac{1}{2}}\sigma_1 d}\right) - F\left(\frac{-k + \frac{2\sigma_1\sigma_2\zeta}{\sigma_1 + \sigma_2} - 2\sigma_1\zeta_2}{2^{\frac{1}{2}}\sigma_1 d}\right)$$

$$= F(b) - F(c) = 1 - \varepsilon$$

$$k = 2^{\frac{1}{2}}\sigma_1 db - \frac{2\sigma_1\sigma_2\zeta_2}{\sigma_1 + \sigma_2} + 2\sigma_1\zeta_2$$

Under H_1:

$$\Pr\left(-k + \frac{2\sigma_1\sigma_2\zeta_2}{\sigma_1 + \sigma_2} < x < k + \frac{2\sigma_1\sigma_2\zeta_2}{\sigma_1 + \sigma_2} \right) = F\left(\frac{k + \dfrac{2\sigma_1\sigma_2\zeta_2}{\sigma_1 + \sigma_2} - 2\sigma_2\zeta_2}{2^{1/2}\sigma_2 d} \right) - F\left(\frac{-k + \dfrac{2\sigma_1\sigma_2\zeta_2}{\sigma_1 + \sigma_2} - 2\sigma_2\zeta_2}{2^{1/2}\sigma_2 d} \right)$$

$$= F\left[\frac{\sigma_1}{\sigma_2} b - \frac{2^{1/2}\zeta_2}{d}\left(1 - \frac{\sigma_1}{\sigma_2} \right) \right] - F\left[\frac{\sigma_1}{\sigma_2} c - \frac{2^{1/2}\zeta_2}{d}\left(1 - \frac{\sigma_1}{\sigma_2} \right) \right] = 1 - P_0$$

Again determining b such that $F(b) = 1 - \varepsilon$, the orders of the two symmetric quantiles which maximize P_0 are

$$p_1 = F(1.483) = .0690$$
$$p_2 = F(-1.483) = .9310$$

For these values, $d = 1.8364/n^{1/2}$ and $2^{1/2}\zeta/d = 1.142\,n^{1/2}$, so that the optimum one-sided power function is given by

$$P_0 = 1 - F\left[\frac{\sigma_1}{\sigma_2} b - 1.142\,n^{1/2}\left(1 - \frac{\sigma_1}{\sigma_2} \right) \right]$$

We are only considering symmetric quantiles because we conjecture, but have not yet been able to prove that this procedure is optimum.

Figure 5 shows the power and efficiency of Test B_2 for $n = 200$, $\varepsilon = 0.1$. For this case, if

$$x = z(.9310) - z(.0690) < \sigma_1\left(2.966 + \frac{6.041}{n^{1/2}} \right)$$

accept H_0. Otherwise reject H_0. The efficiency never drops below .67.
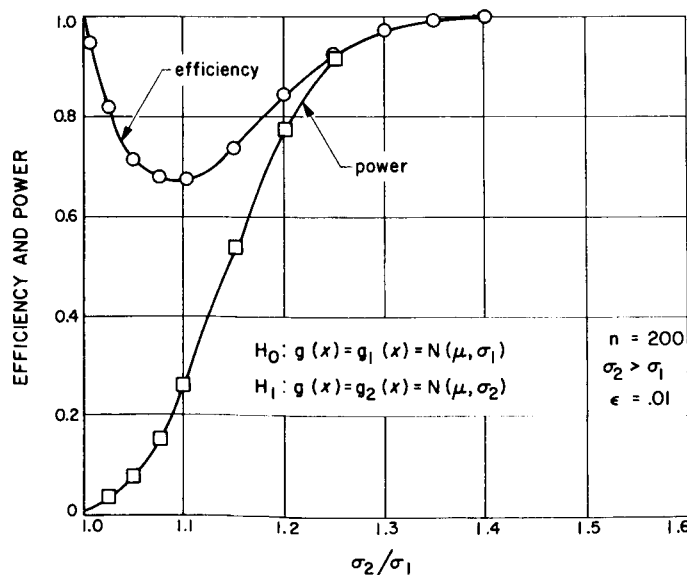
Fig. 5. Power and efficiency of Test B_2 using two quantiles

C. Test B_i: Four Quantiles

For the four-quantile case, let z_i ($i = 1, 2, 3, 4$) be four sample quantiles such that $p_1 + p_4 = p_2 + p_3 = 1$. The test will be based on the linear combination

$$y = \alpha(z_4 - z_1) + \beta(z_3 - z_2)$$

The proportions α and β, as well as the orders of the quantiles, will be determined so as to maximize P_0.

Under H_0:

$$E(y) = 2\sigma_1(\alpha\zeta_4 + \beta\zeta_3)$$

$$\text{Var}(y) = 2\sigma_1^2[\alpha^2 a_3^2 + \beta^2 a_4^2 + 2\alpha\beta\, a_1 a_2(\rho_{12} - \rho_{13})] = 2\sigma_1^2\gamma^2$$

where

$$a_1^2 = \frac{F(\zeta_4)[1 - F(\zeta_4)]}{nf^2(\zeta_4)}$$

$$a_2^2 = \frac{F(\zeta_3)[1 - F(\zeta_3)]}{nf^2(\zeta_3)}$$

$$a_3^2 = \frac{[1 - F(\zeta_4)][2F(\zeta_4) - 1]}{nf^2(\zeta_4)}$$

$$a_4^2 = \frac{[1 - F(\zeta_3)][2F(\zeta_3) - 1]}{nf^2(\zeta_3)}$$

$$\rho_{ij} = \text{the correlation between } z_i \text{ and } z_j$$

Under H_1:

$$E(y) = 2\sigma_2(\alpha\zeta_4 + \beta\zeta_3)$$

$$\text{Var}(y) = 2\sigma_2^2\gamma^2$$

The likelihood ratio inequality provides as the best critical region

$$\left[y - \frac{2\sigma_1\sigma_2(\alpha\zeta_4 + \beta\zeta_3)}{\sigma_1 + \sigma_2}\right]^2 > k^2 \qquad (\sigma_2 > \sigma_1)$$

Moreover,

$$F\left[\frac{k + \dfrac{2\sigma_1\sigma_2(\alpha\zeta_4 + \beta\zeta_3)}{\sigma_1 + \sigma_2} - 2\sigma_1(\alpha\zeta_4 + \beta\zeta_3)}{2^{1/2}\sigma_1\gamma}\right] - F\left[\frac{-k + \dfrac{2\sigma_1\sigma_2(\alpha\zeta_4 + \beta\zeta_3)}{\sigma_1 + \sigma_2} - 2\sigma_1(\alpha\zeta_4 + \beta\zeta_3)}{2^{1/2}\sigma_1\gamma}\right] = F(b) - F(c) = 1 - \varepsilon$$

$$k = 2^{1/2}\sigma_1\gamma b + 2\sigma_1(\alpha\zeta_4 + \beta\zeta_3) - \frac{2\sigma_1\sigma_2(\alpha\zeta_4 + \beta\zeta_3)}{\sigma_1 + \sigma_2}$$

and

$$F\left[\frac{\sigma_1}{\sigma_2}b - \frac{2^{1/2}\left(1 - \dfrac{\sigma_1}{\sigma_2}\right)(\alpha\zeta_4 + \beta\zeta_3)}{\gamma}\right] - F\left[\frac{\sigma_1}{\sigma_2}c - \frac{2^{1/2}\left(1 - \dfrac{\sigma_1}{\sigma_2}\right)(\alpha\zeta_4 + \beta\zeta_3)}{\gamma}\right] = 1 - P_0$$

As in the previous cases, we determine b such that $F(b) = 1 - \varepsilon$. From previous investigations in the use of quantiles in estimating the parameters of normal distributions (Ref. 1, p. 7), it can be deduced that for

$$\alpha = .116 \qquad \beta = .236$$

$$p_1 = .023 \qquad p_2 = .127$$

$$p_4 = .977 \qquad p_3 = .873$$

P_0 will be a maximum. Hence the power function is given by

$$P_0 = 1 - F\left[\frac{\sigma_1}{\sigma_2} b - 1.284 \, n^{1/2} \left(1 - \frac{\sigma_1}{\sigma_2}\right)\right]$$
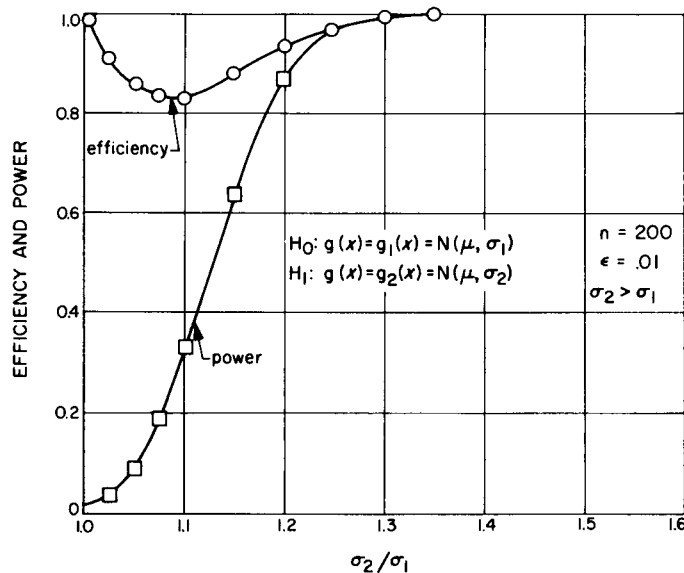
Fig. 6. Power and efficiency of Test B_4 using four quantiles

Figure 6 shows the power and efficiency of Test B_4 for $n = 200$, $\varepsilon = .01$. For this case, if

$$y = .116 \left[z(.977) - z(.023)\right] + .236 \left[z(.873) - z(.127)\right] < \sigma_1 \left(1.001 + \frac{1.814}{n^{1/2}}\right)$$

accept H_0. Otherwise reject H_0. The efficiency for the four-quantile case never drops below .83.

If $\sigma_1 > \sigma_2$ is specified instead of $\sigma_2 > \sigma_1$, a similar analysis provides the following *acceptance* regions. If

$$z(.9424) > \mu - \sigma_1 \left(1.575 - \frac{4.697}{n^{1/2}}\right)$$

$$z(.0576) < \mu + \sigma_1 \left(1.575 - \frac{4.697}{n^{1/2}}\right)$$

$$z(.9310) - z(.0690) > \sigma_1 \left(2.966 - \frac{6.041}{n}\right)$$

$$.116 \left[z(.977) - z(.023) \right] + .236 \left[z(.873) - z(.127) \right] > \sigma_1 \left(1.001 - \frac{1.814}{n^{1/2}} \right)$$

accept H_0. Otherwise reject H_0.

If it is not known whether σ_2 is greater or less than σ_1, in which case H_1 is no longer simple but composite, the test can be modified slightly and the power will, of necessity, be reduced. This remark also holds for Test A if it is not known whether μ_2 is greater or less than μ_1. Details are omitted.

V. TEST C: TESTING THE MEAN AND STANDARD DEVIATION OF A NORMAL DISTRIBUTION

In the use of sample quantiles for estimation and prediction, the orders of the quantiles chosen for the task are of crucial importance because the efficiencies of the estimators and tests depend sharply upon their choice. In fact, although a small deviation from the optimum is of little consequence, the loss in efficiency resulting from a more or less random selection can be quite large. For example, in Test A using two quantiles, the efficiency for $\mu_2 - \mu_1 / \sigma = .1$ is .8077 when the optimal quantiles, those of orders $p_1 = .2703$ and $p_2 = .7297$, are used. If, however, one uses, say, the optimal quantiles for Test B, those of orders $p_1 = .0690$ and $p_2 = .9310$, the efficiency drops to .5210, a loss in efficiency of 35.5%; whereas for, say, $p_1 = .26$ and $p_2 = .74$, the efficiency is .8060, a loss in efficiency of only 2.1%. Thus the desirability of optimal quantiles is far from being academic.

In each of the two previous tests, one of the two parameters is assumed to be the same for both of the hypothesized distributions, and the test is performed with respect to the other. Consequently, since estimation and prediction are often closely related, it was found that the quantiles that should be used to *estimate* the mean are also those that should be used to *test* the mean, and similarly for the one-sided tests of the standard deviation. In fact, since the optimal choices of up to 10 pairs of symmetric quantiles have already been determined for the estimation problem and are given in Ref. 1, it would be a relatively simple task to devise test statistics using a like number of quantiles for both Test A and Test B; but since the minimum efficiency using only four quantiles is

already high in both cases, this procedure has been omitted.

With respect to the present test, however, no such prior knowledge is at hand. It should be intuitively obvious that neither the sets of quantiles used in Test A nor those used in Test B should be used for Test C. The power function for the one-quantile case verifies this conjecture and also shows, as we shall see, that the order of the optimum single quantile depends on the ratios σ_2 / σ_1 and $\mu_2 - \mu_1 / \sigma_2$ even when, as in Test B, we substitute for simplicity the best one-sided test for the best two-sided test. Up to the present time, only the optimum single quantile has been determined for the modified Test C for various values of the above ratios. A preliminary investigation into the two-quantile case shows, unfortunately, that the use of symmetric quantiles is a poor procedure. Were this not so, the task of finding the optimum pair of quantiles would be relatively simple. The investigation also shows that unless optimal or nearly optimal quantiles are used, the advantage to be gained over a single optimum quantile is likely to be small. Consequently, only the one-quantile case will be discussed here.

If one wishes to decide whether the parameters of a normal distribution are μ_1 and σ_1 or μ_2 and σ_2, one can make a decision concerning the "truth" of one or the other of the following two simple hypotheses.

$$H_0: \quad g(x) = g_1(x) = N(\mu_1, \sigma_1)$$
$$H_1: \quad g(x) = g_2(x) = N(\mu_2, \sigma_2)$$

Although neither the true mean nor the true variance is known, the hypotheses are simple, because it is assumed that one has some reason to believe that the parameters of $g(x)$ are *either* the set (μ_1, σ_1) *or* the set (μ_2, σ_2). The test will again be one-sided and will be based on the value of a single quantile. The power functions P_0 and P'_0 will be derived and the efficiency of the test determined as a function of the ratios σ_2/σ_1 and $\mu_2 - \mu_1/\sigma_2$.

Accordingly, let z be the sample quantile of order p. The likelihood ratio inequality applied to z gives

$$\frac{(z - \sigma_2\zeta - \mu_2)^2}{\sigma_2^2} - \frac{(z - \sigma_1\zeta - \mu_1)^2}{\sigma_1^2} < c_1$$

$$(\sigma_1^2 - \sigma_2^2)z^2 - 2z(\sigma_1^2\sigma_2\zeta + \sigma_1^2\mu_2 - \sigma_1\sigma_2^2\zeta - \sigma_2^2\mu_1) < c_2$$

$$\left[z - \frac{\sigma_1\sigma_2\zeta(\sigma_1 - \sigma_2) + \sigma_1^2\mu_2 - \sigma_2^2\mu_1}{\sigma_1^2 - \sigma_2^2} \right]^2 = (z - \alpha)^2 > k^2 \qquad (\sigma_2 > \sigma_1)$$

For $\sigma_2 > \sigma_1$, one has

Under H_0:

$$\Pr(-k + \alpha < z < k + \alpha) = F\left(\frac{k + \alpha - \sigma_1\zeta - \mu_1}{\sigma_1 a}\right) - F\left(\frac{-k + \alpha - \sigma_1\zeta - \mu_1}{\sigma_1 a}\right)$$

$$= F\left[\frac{k}{\sigma_1 a} + \frac{\zeta\left(1 - \frac{\sigma_1}{\sigma_2}\right) + \frac{\mu_2 - \mu_1}{\sigma_2}}{a\left(\frac{\sigma_1}{\sigma_2} - \frac{\sigma_2}{\sigma_1}\right)}\right] - F\left[\frac{-k}{\sigma_1 a} + \frac{\zeta\left(1 - \frac{\sigma_1}{\sigma_2}\right) + \frac{\mu_2 - \mu_1}{\sigma_2}}{a\left(\frac{\sigma_1}{\sigma_2} - \frac{\sigma_2}{\sigma_1}\right)}\right]$$

$$= F(b) - F(c) = 1 - \varepsilon.$$

where

$$k = \sigma_1 ab + \sigma_1\zeta + \mu_1 - \alpha$$

and

$$a^2 = \frac{F(\zeta)[1 - F(\zeta)]}{nf^2(\zeta)}$$

Under H_1:

$$\Pr[-k + \alpha < z < k + \alpha] = F\left[\frac{k + \alpha - \sigma_2 - \mu_2}{\sigma_2 a}\right] - F\left[\frac{-k + \alpha - \sigma_2 - \mu_2}{\sigma_2 a}\right]$$

$$= F\left[\frac{\sigma_1}{\sigma_2}b - \frac{\zeta\left(1 - \frac{\sigma_1}{\sigma_2}\right) + \frac{\mu_2 - \mu_1}{\sigma_2}}{a}\right] - F\left[\frac{\sigma_1}{\sigma_2}c - \frac{\zeta\left(1 - \frac{\sigma_1}{\sigma_2}\right) + \frac{\mu_2 - \mu_1}{\sigma_2}}{a}\right] = 1 - P_0$$

$$(7)$$

As in Part III, we shall use the best one-sided test instead of the above two-sided test and shall determine the order of z which maximizes its power. However, for this test it is necessary to distinguish two cases:

1. $\mu_2 - \mu_1 > 0$

2. $\mu_2 - \mu_1 < 0$

The necessity for making this distinction arises from the fact that the form of the acceptance region for the one-sided test depends upon the relative values of μ_1 and μ_2 for a fixed ratio σ_2/σ_1. Since

$$E(z \,|\, H_0) = \sigma_1 \zeta + \mu_1$$

$$E(z \,|\, H_1) = \sigma_2 \zeta + \mu_2$$

one sees that for $\sigma_2/\sigma_1 > 1$ and $\mu_2 - \mu_2 > 0$, one should choose $\zeta > 0$ (p > .5) in order to separate as much as possible the means of the density functions of z under H_0 and H_1, say $h_1(z \,|\, H_0)$ and $h_2(z \,|\, H_1)$, respectively, thus maximizing P_0. The acceptance region of the two-sided test is of the form

$$\delta_2 < z < \delta_1$$

where

$$\Pr(\delta_2 < z \,|\, H_0 < \delta_1) = 1 - \varepsilon$$

Since in this case $h_2(z \,|\, H_1)$ will be to the right of $h_1(z \,|\, H_0)$, the best one-sided acceptance region will be obtained by putting $\zeta_2 = -\infty$, which results in

$$z < \delta_1$$

We determine δ_1 such that

$$\Pr(z \,|\, H_0 < \delta_1) = 1 - \varepsilon$$

On the other hand, for $\sigma_2/\sigma_1 > 1$ and $\mu_2 - \mu_1 < 0$, one should choose $\zeta < 0$ (p < .5) so that $h_2(z \,|\, H_1)$ will be as far to the left of $h_1(z \,|\, H_0)$ as possible. The best one-sided acceptance region for this case is obtained by putting $\delta_1 = \infty$, which results in

$$\delta_2 < z$$

We determine δ_2 such that

$$\Pr(\delta_2 < z \,|\, H_0) = 1 - \Pr(z \,|\, H_0 < \delta_2) = 1 - \varepsilon$$

or

$$\Pr(z \,|\, H_0 < \delta_2) = \varepsilon$$

In view of the above analysis, for the one-sided test, Eq. 7 for $\sigma_2/\sigma_1 > 1$ becomes, under each of the two conditions

$$F\left[\frac{\sigma_1}{\sigma_2} b_1 - \frac{\zeta\left(1 - \dfrac{\sigma_1}{\sigma_2}\right) + \dfrac{\mu_2 - \mu_1}{\sigma_2}}{a}\right] = 1 - P_0 \qquad (\mu_2 - \mu_1 > 0) \tag{8}$$

where

$$F(b_1) = 1 - \varepsilon$$

and

$$F\left[\frac{\sigma_1}{\sigma_2}c_1 - \frac{\zeta\left(1 - \frac{\sigma_1}{\sigma_2}\right) + \frac{\mu_2 - \mu_1}{\sigma_2}}{a}\right] = P_0 \qquad (\mu_2 - \mu_1 < 0) \qquad (9)$$

where

$$F(c_1) = F(-b_1) = \varepsilon$$

It is evident from Eq. 8 and 9 that in order to maximize P_0, the order of z should be chosen to maximize the quantity $A\zeta + B/a$ in Eq. 8 and minimize $A\zeta + B/a$ in Eq. 9, where

$$A = 1 - \frac{\sigma_1}{\sigma_2}$$

$$B = \frac{\mu_2 - \mu_1}{\sigma_2}$$

Putting

$$h(\zeta) = \left(\frac{A\zeta + B}{a}\right)^2 = \frac{nf^2(A\zeta + B)^2}{F(1 - F)}$$

where

$$f = f(\zeta) \quad \text{and} \quad F = F(\zeta)$$

and setting $h'(\zeta)$ equal to zero, one has

$$2F(1 - F)[A - \zeta(A\zeta + B)] - f(A\zeta + B)[1 - 2F]$$

$$= 2F(1 - F)\left\{1 - \frac{\sigma_1}{\sigma_2} - \zeta\left[\zeta\left(1 - \frac{\sigma_1}{\sigma_2}\right) + \frac{\mu_2 - \mu_1}{\sigma_2}\right]\right\} - \left[\zeta\left(1 - \frac{\sigma_1}{\sigma_2}\right) + \frac{\mu_2 - \mu_1}{\sigma_2}\right][f(1 - 2F)] = 0 \qquad (10)$$

Now suppose that, for a given value of $\mu_2 - \mu_1$, $\bar{\zeta}$ satisfies Eq. 10, that is, $h'(\bar{\zeta}) = 0$. Then $h'(-\bar{\zeta}) = 0$ for $\mu'_2 - \mu'_1 = -(\mu_2 - \mu_1)$. For, evaluating Eq. 10 for $-\bar{\zeta}$ and $\mu'_2 - \mu'_1$, and noting that $F(-\bar{\zeta}) = 1 - F(\bar{\zeta})$ and $f(-\bar{\zeta}) = f(\bar{\zeta})$, one obtains

$$2F(1 - F)\left\{1 - \frac{\sigma_1}{\sigma_2} + \bar{\zeta}\left[-\bar{\zeta}\left(1 - \frac{\sigma_1}{\sigma_2}\right) - \frac{\mu_2 - \mu_1}{\sigma_2}\right]\right\}$$

$$- \left[-\bar{\zeta}\left(1 - \frac{\sigma_1}{\sigma_2}\right) - \frac{\mu_2 - \mu_1}{\sigma_2}\right][f(2F - 1)] = 0$$

Thus for a given value of $\mu_2 - \mu_1 > 0$, that *positive* value of ζ which satisfies Eq. 10 should also be used in Eq. 8, and $-\zeta$ should be used in Eq. 9 for $\mu'_2 - \mu'_1 = -(\mu_2 - \mu_1)$. In other words, if the quantile of order $p > .5$ maximizes P_0 in Eq. 8 for $\mu_2 - \mu_1 > 0$, then the quantile of order $p' = 1 - p$ will maximize P_0 in Eq. 9 for $-(\mu_2 - \mu_1) < 0$.

The optimum acceptance regions for these cases are of the form

$$z(p) < \mu_1 + \sigma_1(\mathbf{ab}_1 + \zeta) \qquad (\mu_2 > \mu_1, \sigma_2 > \sigma_1)$$

$$z(p') > \mu_1 + \sigma_1(\mathbf{a}'\mathbf{c}_1 + \zeta') \qquad (\mu_2 < \mu_1, \sigma_2 > \sigma_1)$$

where

$$\zeta > 0, \quad F(\zeta) = p > .5, \quad F(b_1) = 1 - \varepsilon$$

$$\zeta' < 0, \quad F(\zeta') = p' < .5, \quad F(c_1) = F(-b_1) = \varepsilon$$

$$a^2 = \frac{F(\zeta)[1 - F(\zeta)]}{nf^2(\zeta)}$$

$$(a')^2 = \frac{F(\zeta')[1 - F(\zeta')]}{nf^2(\zeta')}$$

A similar analysis for $\sigma_2/\sigma_1 < 1$ gives the following optimum acceptance regions. The details are omitted.

$$z(\widetilde{p}) < \mu_1 + \sigma_1(\widetilde{\mathbf{db}} + \widetilde{\zeta}) \qquad (\mu_2 > \mu_1, \sigma_2 < \sigma_1)$$

$$z(\widetilde{p'}) > \mu_1 + \sigma_1(\widetilde{\mathbf{d'c}} + \widetilde{\zeta'}) \qquad (\mu_2 < \mu_1, \sigma_2 < \sigma_1)$$

where

$$\widetilde{\zeta} < 0, \quad F(\widetilde{\zeta}) = \widetilde{p} < .5, \quad F(\widetilde{b}) = 1 - \varepsilon$$

$$\widetilde{\zeta'} > 0, \quad F(\widetilde{\zeta'}) = \widetilde{p'} > .5, \quad F(\widetilde{c}) = 1 - F(\widetilde{b}) = \varepsilon$$

$$d^2 = \frac{F(\widetilde{\zeta})[1 - F(\widetilde{\zeta})]}{nf^2(\widetilde{\zeta})}$$

$$(d')^2 = \frac{F(\widetilde{\zeta'})[1 - F(\widetilde{\zeta'})]}{nf^2(\widetilde{\zeta'})}$$

The likelihood ratio inequality in the case of the entire sample gives, as the best critical region,

$$y = \sum_{i=1}^{n} (x_i - \bar{a})^2 > k \qquad (\sigma_2 > \sigma_1)$$

where

$$\bar{a} = \frac{\mu_1 \sigma_2^2 - \mu_2 \sigma_1^2}{\sigma_2^2 - \sigma_1^2}$$

Since the Central Limit Theorem assures us that y is approximately normal for $n \geq 200$, the limiting distribution and moments are assumed. Hence

Under H_0:

$$E(y) = n[\sigma_1^2 + (\mu_1 - \bar{a})^2]$$

$$\text{Var}(y) = n[2\sigma_1^4 + 4\sigma_1^2(\mu_1 - \bar{a})^2]$$

$$\Pr(y < k) = F\left\{\frac{k - n[\sigma_1^2 + (\mu_1 - \bar{a})^2]}{\sqrt{n[2\sigma_1^4 + 4\sigma_1^2(\mu_1 - \bar{a})^2]}}\right\} = F(b) = 1 - \varepsilon$$

$$k = b\sqrt{n[2\sigma_1^4 + 4\sigma_1^2(\mu_1 - \bar{a})^2]} + n[\sigma_1^2 + (\mu_1 - \bar{a})^2]$$

Under H_1:

$$E(y) = n\left[\sigma_2^2 + (\mu_2 - \bar{\alpha})^2\right]$$

$$\mathrm{Var}(y) = n\left[2\sigma_2^4 + 4\sigma_2^2(\mu_2 - \bar{\alpha})^2\right]$$

$$\Pr(y < k) = F\left\{\frac{k - n\left[\sigma_2^2 + (\mu_2 - \bar{\alpha})^2\right]}{\sqrt{n\left[2\sigma_2^4 + 4\sigma_2^2(\mu_2 - \bar{\alpha})^2\right]}}\right\}$$

$$= F\left\{\frac{b\sqrt{n\left[2\sigma_1^4 + 4\sigma_1^2(\mu_1 - \bar{\alpha})^2\right]} + n\left[\sigma_1^2 + (\mu_1 - \bar{\alpha})^2\right] - n\left[\sigma_2^2 + (\mu_2 - \bar{\alpha})^2\right]}{\sqrt{n\left[2\sigma_2^4 + 4\sigma_2^2(\mu_2 - \bar{\alpha})^2\right]}}\right\}$$

$$= F\left\{\frac{\sigma_1^2}{\sigma_2^2}\, b\, \sqrt{\frac{\left(1 - \frac{\sigma_1^2}{\sigma_2^2}\right)^2 + 2\frac{\sigma_1^2}{\sigma_2^2}\left(\frac{\mu_2 - \mu_1}{\sigma_2}\right)^2}{\left(1 - \frac{\sigma_1^2}{\sigma_2^2}\right)^2 + 2\left(\frac{\mu_2 - \mu_1}{\sigma_2}\right)^2}} - \frac{\sqrt{\frac{n}{2}}\left(1 - \frac{\sigma_1^2}{\sigma_2^2}\right)^2 + \left(1 + \frac{\sigma_1^2}{\sigma_2^2}\right)\left(\frac{\mu_2 - \mu_1}{\sigma_2}\right)^2}{\sqrt{\left(1 - \frac{\sigma_1^2}{\sigma_2^2}\right)^2 + 2\left(\frac{\mu_2 - \mu_1}{\sigma_2}\right)^2}}\right\} = 1 - P_0'$$

Table 1a. Optimum test statistics to be used in the one-sided Test C for $\sigma_2/\sigma_1 > 1$

			If $p > .5$, accept H_0 if $z(p) < \delta$ If $p < .5$, accept H_0 if $z(p) > -\delta$ $\dfrac{\delta - \mu_1}{\sigma_1}$				
σ_2/σ_1	$\dfrac{\mu_2 - \mu_1}{\sigma_2}$	Opt p	$\varepsilon = .01$	$\varepsilon = .05$	P_0	P'_0	Eff
1.025	±.05	.8133 (.1867)	$.89 + 3.376/n^{\frac{1}{2}}$	$.89 + 2.387/n^{\frac{1}{2}}$	.0582	.0909	.6403
1.025	±.10	.7190 (.2810)	$.58 + 3.101/n^{\frac{1}{2}}$	$.58 + 2.193/n^{\frac{1}{2}}$	.1450	.2408	.6022
1.025	±.15	.6628 (.3372)	$.42 + 3.010/n^{\frac{1}{2}}$	$.42 + 2.129/n^{\frac{1}{2}}$	.3056	.4852	.6022
1.025	±.20	.6255 (.3745)	$.32 + 2.970/n^{\frac{1}{2}}$	$.32 + 2.101/n^{\frac{1}{2}}$	.5127	.7391	.6936
1.025	±.25	.6026 (.3974)	$.26 + 2.951/n^{\frac{1}{2}}$	$.26 + 2.087/n^{\frac{1}{2}}$	.7217	.9071	.7956
1.025	±.30	.5871 (.4129)	$.22 + 2.941/n^{\frac{1}{2}}$	$.22 + 2.080/n^{\frac{1}{2}}$	.8741	.9777	.8940
1.025	±.35	.5754 (.4246)	$.19 + 2.935/n^{\frac{1}{2}}$	$.19 + 2.075/n^{\frac{1}{2}}$	.9560	1.0	.9560
1.05	±.05	.8790 (.1210)	$1.17 + 3.770/n^{\frac{1}{2}}$	$1.17 + 2.666/n^{\frac{1}{2}}$	.0980	.1774	.5524
1.05	±.10	.8133 (.1867)	$.89 + 3.376/n^{\frac{1}{2}}$	$.89 + 2.387/n^{\frac{1}{2}}$	.2139	.3436	.6225
1.05	±.15	.7580 (.2420)	$.70 + 3.190/n^{\frac{1}{2}}$	$.70 + 2.256/n^{\frac{1}{2}}$	.3732	.5793	.6442
1.05	±.20	.7158 (.2842)	$.57 + 3.094/n^{\frac{1}{2}}$	$.57 + 2.188/n^{\frac{1}{2}}$	.5775	.7976	.7240
1.05	±.25	.6844 (.3156)	$.48 + 3.041/n^{\frac{1}{2}}$	$.48 + 2.151/n^{\frac{1}{2}}$	.7692	.9307	.8265
1.05	±.30	.6591 (.3409)	$.41 + 3.006/m^{\frac{1}{2}}$	$.41 + 2.126/n^{\frac{1}{2}}$	.9000	.9837	.9149
1.05	±.35	.6368 (.3632)	$.35 + 2.982/n^{\frac{1}{2}}$	$.35 + 2.109/n^{\frac{1}{2}}$	.9667	1.0	.9667
1.10	±.05	.9115 (.0885)	$1.35 + 4.119/n^{\frac{1}{2}}$	$1.35 + 2.913/n^{\frac{1}{2}}$	.2312	.4811	.4805
1.10	±.10	.8749 (.1251)	$1.15 + 3.737/n^{\frac{1}{2}}$	$1.15 + 2.643/n^{\frac{1}{2}}$	.3766	.6205	.6069
1.10	±.15	.8389 (.1611)	$.99 + 3.503/n^{\frac{1}{2}}$	$.99 + 2.477/n^{\frac{1}{2}}$	.5564	.7820	.7115
1.10	±.20	.8051 (.1949)	$.86 + 3.343/n^{\frac{1}{2}}$	$.86 + 2.364/n^{\frac{1}{2}}$	.7332	.9053	.8099
1.10	±.30	.7518 (.2482)	$.68 + 3.174/n^{\frac{1}{2}}$	$.68 + 2.244/n^{\frac{1}{2}}$	.9498	.9932	.9546
1.10	±.35	.7291 (.2709)	$.61 + 3.121/n^{\frac{1}{2}}$	$.61 + 2.207/n^{\frac{1}{2}}$	.9845	1.0	.9845
1.15	±.05	.9207 (.0793)	$1.41 + 4.258/n^{\frac{1}{2}}$	$1.41 + 3.012/n^{\frac{1}{2}}$	.4146	.7879	.5261
1.15	±.10	.8962 (.1038)	$1.26 + 3.933/n^{\frac{1}{2}}$	$1.26 \pm 2.781/n^{\frac{1}{2}}$	.5748	.8483	.6776
1.15	±.20	.8461 (.1539)	$1.02 + 3.540/n^{\frac{1}{2}}$	$1.02 + 2.504/n^{\frac{1}{2}}$	.8581	.9673	.8871
1.15	±.30	.7996 (.2004)	$.84 + 3.332/n^{\frac{1}{2}}$	$.84 + 2.357/n^{\frac{1}{2}}$	.9790	1.0	.9790
1.20	±.05	.9251 (.0749)	$1.44 + 4.327/n^{\frac{1}{2}}$	$1.44 + 3.060/n^{\frac{1}{2}}$	.6050	.9344	.6475
1.20	±.10	.9066 (.0934)	$1.32 + 4.056/n^{\frac{1}{2}}$	$1.32 + 2.868/n^{\frac{1}{2}}$	.7446	.9559	.7790
1.20	±.20	.8686 (.1314)	$1.12 + 3.688/n^{\frac{1}{2}}$	$1.12 + 2.608/n^{\frac{1}{2}}$	.8686	.9912	.9429
1.20	±.25	.8485 (.1515)	$1.03 + 3.554/n^{\frac{1}{2}}$	$1.03 + 2.513/n^{\frac{1}{2}}$	.9753	1.0	.9753

Table 1a gives the *optimum* orders of the test statistics and acceptance regions for various values of $\sigma_2/\sigma_1 > 1$ and $\pm \mu_2 - \mu_1/\sigma_2$, for $\varepsilon = .01$ and $\varepsilon = .05$. The third column headed "Opt p" gives the orders of the quantiles to be used in the test. Each entry in this column should be interpreted to mean that the larger value of p, that on the left, should be used for $\mu_2 - \mu_1 > 0$ and the smaller value, that on the right, for $\mu_2 - \mu_1 < 0$. It should be emphasized, however, that although a given value of p in the table is, for the given significance level, the order of *the* test statistic $z(p)$ which maximizes the power of the test when used in conjunction with the given values of σ_2/σ_1 and $\mu_2 - \mu_1/\sigma_2$ shown in the same row, nevertheless this same value of p is the order of *a* test statistic, at the same significance level, for *any* set of ratios $\sigma_2/\sigma_1 > 1$ and $\mu_2 - \mu_1/\sigma_2$ of the same parity, so long as the same value of δ associated with it in the table is used. A suboptimal use of the test statistics affects only the power and not the significance level. This follows from the fact that the acceptance regions do not depend

upon the parameters of $g_2(x)$. If, for example, $\sigma_2/\sigma_1 = 1.12$, $\mu_2 - \mu_1/\sigma_2 = -.06$, one can use the optimum value of $p = .0885$ for $\sigma_2/\sigma_1 = 1.10$, $\mu_2 - \mu_1/\sigma_2 = -.05$, and the test, for $\varepsilon = .01$, can be stated as follows: if

$$z(.0885) > - \left[\mu_1 + \sigma_1 \left(1.35 + \frac{4.119}{n^{1/2}} \right) \right]$$

accept H_0. Otherwise reject H_0. The probability of rejecting H_0 when H_0 is true still remains at .01, while the probability of rejecting H_0 when H_0 is false should not differ much from .2312.

Table 1b gives the same information for $\sigma_2/\sigma_1 < 1$ as that contained in Table 1a. Here, values of $p < .5$ are associated with $\mu_2 - \mu_1 > 0$, and values of $p > .5$ are associated with $\mu_2 - \mu_1 < 0$. In both tables, the values of P_0, P_0', and $\text{Eff} = P_0/P_0'$ were computed for $n = 200$, $\varepsilon = .01$.

Table 1b. Optimum test statistics to be used in the one-sided Test C for $\sigma_2/\sigma_1 < 1$

			If $p < .5$, accept H_0 if $z(p) < -\delta$ If $p > .5$, accept H_0 if $z(p) > \delta$ $\dfrac{\delta - \mu_1}{\sigma_1}$				
σ_2/σ_1	$\dfrac{\mu_2 - \mu_1}{\sigma_2}$	Opt p	$\varepsilon = .01$	$\varepsilon = .05$	P_0	P_0'	Eff
.95	±.05	.1170 (.8830)	$1.19 - 3.805/n^{1/2}$	$1.19 - 2.691/n^{1/2}$	.0702	.0957	.7335
.95	±.10	.1762 (.8238)	$.93 - 3.423/n^{1/2}$	$.93 - 2.421/n^{1/2}$	.1546	.2151	.7187
.95	±.20	.2709 (.7291)	$.61 - 3.121/n^{1/2}$	$.61 - 2.207/n^{1/2}$	.4991	.6824	.7314
.95	±.30	.3300 (.6700)	$.44 - 3.021/n^{1/2}$	$.44 - 2.136/n^{1/2}$	.8579	.9710	.8835
.95	±.40	.3669 (.6331)	$.34 - 2.978/n^{1/2}$	$.34 - 2.106/n^{1/2}$	.9849	1.0	.9849
.90	±.05	.0838 (.9162)	$1.38 - 4.185/n^{1/2}$	$1.38 - 2.960/n^{1/2}$	.1620	.3294	.4918
.90	±.10	.1112 (.8888)	$1.22 - 3.859/n^{1/2}$	$1.22 - 2.729/n^{1/2}$	.2822	.4462	.6325
.90	±.20	.1685 (.8315)	$.96 - 3.461/n^{1/2}$	$.96 - 2.447/n^{1/2}$	.6296	.9530	.6607
.90	±.30	.2206 (.7794)	$.77 - 3.252/n^{1/2}$	$.77 - 2.300/n^{1/2}$	.9064	.9998	.9066
.90	±.35	.2420 (.7580)	$.70 - 3.190/n^{1/2}$	$.70 - 2.256/n^{1/2}$	.9662	1.0	.9662
.80	±.05	.0681 (.9319)	$1.49 - 4.456/n^{1/2}$	$1.49 - 3.151/n^{1/2}$	.5833	.9795	.5955
.80	±.10	.0808 (.9192)	$1.40 - 4.235/n^{1/2}$	$1.40 - 2.995/n^{1/2}$	.7218	.9862	.7319
.80	±.20	.1056 (.8944)	$1.25 - 3.913/n^{1/2}$	$1.25 - 2.767/n^{1/2}$	.9199	1.0	.9199
.80	±.30	.1314 (.8686)	$1.12 - 3.687/n^{1/2}$	$1.12 - 2.608/n^{1/2}$	.9882	1.0	.9882
.75	±.05	.0655 (.9345)	$1.51 - 4.510/n^{1/2}$	$1.51 - 3.190/n^{1/2}$	.8250	1.0	.8250
.75	±.10	.0749 (.9251)	$1.44 - 4.327/n^{1/2}$	$1.44 - 3.060/n^{1/2}$	.9045	1.0	.9045
.75	±.15	.0838 (.9162)	$1.38 - 4.185/n^{1/2}$	$1.38 - 2.960/n^{1/2}$	.9548	1.0	.9548

VI. TESTS D AND E: TWO SAMPLE TESTS

A. Statement of the Problem

Heretofore, it was assumed that we were given a set of independent sample values taken from a single normally distributed population, and the problem was to decide on the basis of one or more sample quantiles whether the parameters of the distribution had one or another set of values. In this section, however, it is assumed that we are given sets of independent samples taken from *two* independent normally distributed populations with density functions $g_1(x)$ and $g_2(y)$, and we consider the following two tests.

Test D:

$$H_0: g_1(x) = N(\mu, \sigma), g_2(y) = N(\mu, \sigma)$$

$$H_1: g_1(x) = N(\mu, \sigma), g_2(y) = N(\mu + \theta, \sigma), \theta \neq 0$$

where σ is known and μ is not known.

Test E':

$$H_0: g_1(x) = N(\mu_1, \sigma), g_2(y) = N(\mu_2, \sigma)$$

$$H_1: g_1(x) = N(\mu_1, \sigma), g_2(y) = N(\mu_2, \theta\sigma), \theta > 0$$

where μ_1 and μ_2 are known and σ is not known. Since μ_1 and μ_2 are assumed known in Test E', without loss of generality we can put $\mu_1 = \mu_2 = 0$, so that one has

Test E:

$$H_0: g_1(x) = N(0, \sigma), g_2(y) = N(0, \sigma)$$

$$H_1: g_1(x) = N(0, \sigma), g_2(y) = N(0, \theta\sigma), \theta > 0$$

where σ is not known.

For both Test D and Test E, tests are devised using one, two, and four pairs of sample quantiles, one of each pair taken from each of the two distributions. Sample sizes n_1 and n_2 are assumed, where n_1 and n_2 are large ($\rightleftharpoons 200$). In each case, the power function is derived and the efficiency is determined.

B. Test D_1: One Pair of Quantiles

Beginning with one pair of quantiles, let z be the sample quantile (of order p) of the samples taken from the first population, which has a density function $g_1(x)$, and let z' be the corresponding sample quantile (of order p) of the samples taken from the second population. Then one has

Under H_0:

$$E(z) = E(z') = \sigma\zeta + \mu$$

$$Var(z_1) = \frac{\sigma^2 a^2}{n_1}$$

$$Var(z') = \frac{\sigma^2 a^2}{n_2}$$

where

$$a^2 = \frac{F(\zeta)[1 - F(\zeta)]}{f^2(\zeta)}$$

Under H_1:

$$E(z) = \sigma\zeta + \mu$$

$$E(z') = \sigma\zeta + \mu + \theta$$

$$Var(z) = \frac{\sigma^2 a^2}{n_1}$$

$$Var(z') = \frac{\sigma^2 a^2}{n_2}$$

Since μ is unknown, the distribution of the test statistic cannot depend on μ. Hence, the test statistic to be used should be given by

$$w = z - z'$$

Under H_0:

$$E(w) = 0$$

$$Var(w) = \sigma^2 a^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right) = \sigma_w^2$$

Under H_1:

$$E(w) = -\theta$$

$$Var(w) = \sigma_w^2$$

The best critical region is again determined by the likelihood ratio inequality

$$\frac{L(w|H_0)}{L(w|H_1)} = \frac{\exp - \left[\frac{1}{2}\left(\frac{w^2}{\sigma_w^2}\right)\right]}{\exp - \left[\frac{1}{2}\left(\frac{(w+\theta)^2}{\sigma_w^2}\right)\right]} < c$$

which reduces to the region in which

$$\begin{cases} w < k_1, & \theta > 0 \\ w > k_2, & \theta < 0 \end{cases}$$

Now, assuming $\theta > 0$,

Under H_0:

$$\Pr(w < k_1) = F\left(\frac{k_1}{\sigma_w}\right) = F(b) = \varepsilon$$

$$k_1 = \sigma_w b$$

Under H_1:

$$\Pr(w < k) = F\left(\frac{k_1 + \theta}{\sigma_w}\right)$$

$$= F\left(b + \frac{\theta}{\sigma a}\left(\frac{n_1 n_2}{n_1 + n_2}\right)^{1/2}\right) = P_0$$

As in Test A, P_0 depends only upon the quantity a, which in turn depends only upon the order of the pair of quantiles chosen for the test. Again choosing the median, the quantile of order .5, thus maximizing P_0, the power function becomes

$$P_0 = F\left(b + .798\frac{\theta}{\sigma}\left(\frac{n_1 n_2}{n_1 + n_2}\right)^{1/2}\right)$$

To determine the efficiency of the test, P_0' must be derived. The test statistic based on all the sample values is given by

$$v = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i - \frac{1}{n_2}\sum_{i=1}^{n_2} y_i$$

The best critical region is easily found to be the region for which

$$\begin{cases} v < k_3, & \theta > 0 \\ v > k_4, & \theta < 0 \end{cases}$$

and for $\theta > 0$, P_0' is given by

$$P_0' = F\left(b + \frac{\theta}{\sigma}\left(\frac{n_1 n_2}{n_1 + n_2}\right)^{1/2}\right)$$

If we now take $n_1 = n_2 = n = 200$, $\varepsilon = .01$, then

$$P_0 = F\left(-2.326 + .798\frac{\theta}{\sigma}\left(\frac{n}{2}\right)^{1/2}\right)$$

$$= F\left(-2.326 + 7.98\frac{\theta}{\sigma}\right)$$

$$P_0' = F\left(-2.326 + \frac{\theta}{\sigma}\left(\frac{n}{2}\right)^{1/2}\right)$$

$$= F\left(-2.326 + 10\frac{\theta}{\sigma}\right)$$

Figure 7 shows the power efficiency of this test using one optimal pair of quantiles. Under the given conditions,

$$\sigma_w = 1.253\sigma\left(\frac{n_1 + n_2}{n_1 n_2}\right)^{1/2} = .1253\sigma$$

and

$$k_1 = -2.915\sigma\left(\frac{n_1 + n_2}{n_1 n_2}\right)^{1/2} = -.2915\sigma$$

Thus for $\theta > 0$, if the difference between the median of the samples taken from the first population and the
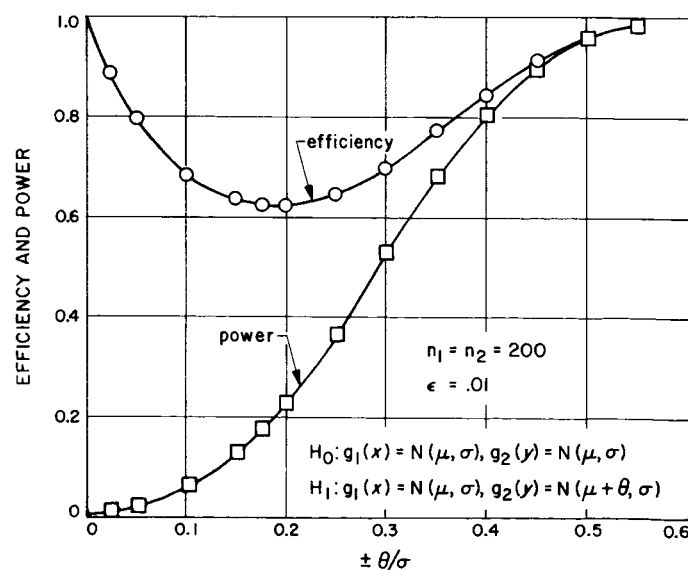

Fig. 7. Power and efficiency of Test D_1 using one pair of quantiles

median of the samples taken from the second population is greater than $-.2914\sigma$, accept H_0. Otherwise reject H_0. That is, if $\theta > 0$ and

$$z(.5) - z'(.5) > -2.915\sigma \left(\frac{n_1 + n_2}{n_1 n_2}\right)^{\frac{1}{2}}$$

accept H_0. Otherwise reject H_0. The test is made at a significance level of .01.

It is interesting to note that as $n_1 \to \infty$ for fixed n_2, the power functions P_0 and P'_0 increase and approach, for $\theta > 0$,

$$P_0 = F\left(-2.326 + .798\frac{\theta}{\sigma}n^{\frac{1}{2}}\right)$$

$$P'_0 = F\left(-2.326 + \frac{\theta}{\sigma}n^{\frac{1}{2}}\right) \qquad (11)$$

These equations (11) are the power functions of Test A_1 which, in the present terminology, are given by

$$H_0: \quad g(x) = g_1(x) = N(\mu, \sigma)$$

$$H_1: \quad g(x) = g_2(x) = N(\mu + \theta, \sigma)$$

where μ and σ are known.

This phenomenon can be explained by the fact that, although μ is unknown in the present test, its value can be estimated exactly with probability one as $n \to \infty$, so that the test is essentially made on the second set of sample values with known μ and σ. This same phenomenon also occurs in the two- and four-quantile cases.

If $\theta < 0$, then

$$w > k_2 = 2.915\sigma$$

is the critical region for the single-pair-of-quantiles test, and

$$P_0 = 1 - F\left(2.326 + .798\frac{\theta}{\sigma}\right)$$

$$P'_0 = 1 - F\left(2.326 + 10\frac{\theta}{\sigma}\right)$$

It is easily seen, therefore, that if $\theta_1 = -\theta_2 < 0$,

$$P_0(\theta_1) = P_0(\theta_2)$$

$$P'_0(\theta_1) = P'_0(\theta_2)$$

For this case, if

$$z(.5) - z'(.5) < 2.915\sigma \left(\frac{n_1 + n_2}{n_1 n_2}\right)^{\frac{1}{2}}$$

accept H_0. Otherwise reject H_0.

C. Test D_2: Two Pairs of Quantiles

Now let z_1 and z_2 be the sample quantiles of the first population, of orders p_1 and $p_2 = 1 - p_1$, and let z'_1 and z'_2 denote the corresponding sample quantiles of the second population. To eliminate μ, let

$$w_1 = z_1 - z'_1$$

$$w_2 = z_2 - z'_2$$

Then one has:

$$E(z_1) = \sigma\zeta_1 + \mu \qquad \text{Var}(z_1) = \frac{\sigma^2 a^2}{n_1}$$

$$E(z_2) = \sigma\zeta_2 + \mu \qquad \text{Var}(z_2) = \text{Var}(z_1)$$
$$= -\sigma\zeta_1 + \mu$$

$$E(z'_1) = \sigma\zeta_1 + \mu \qquad \text{Var}(z'_1) = \frac{\sigma_2 a^2}{n_2}$$

$$E(z'_2) = -\sigma\zeta_1 + \mu \qquad \text{Var}(z'_2) = \text{Var}(z'_1)$$

$$E(w_1) = E(w_2) = 0 \qquad \text{Var}(w_1) = \text{Var}(w_2)$$
$$= \sigma^2 a^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

where

$$a^2 = \frac{F(\zeta_1)[1 - F(\zeta_1)]}{f^2(\zeta_1)}$$

Under H_1:

$$E(z_1) = \sigma\zeta_1 + \mu \qquad \text{Var}(z_1) = \frac{\sigma^2 a^2}{n_1}$$

$$E(z_2) = -\sigma\zeta_1 + \mu \qquad \text{Var}(z_2) = \text{Var}(z_1)$$

$$E(z'_1) = \sigma\zeta_1 + \mu + \theta \qquad \text{Var}(z'_1) = \frac{\sigma^2 a^2}{n_2}$$

$$E(z'_2) = -\sigma\zeta_1 + \mu + \theta \qquad \text{Var}(z'_2) = \text{Var}(z'_1)$$

$$E(w_1) = E(w_2) = -\theta \qquad \text{Var}(w_1) = \text{Var}(w_2)$$
$$= \sigma^2 a^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

The two-quantile test will be made on the statistic

$$y = w_1 + w_2$$

Under H_0:

$$E(y) = 0$$

$$\text{Var}(y) = 2\sigma^2 a^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)(1 + \rho)$$

where ρ is the correlation between z_1 and z_2, and also between z_1' and z_2'.

Under H_1:

$$E(y) = -2\theta$$

$$\text{Var}(y) = 2\sigma^2 a^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)(1 + \rho)$$

From the likelihood ratio inequality, the critical regions are easily determined to be the regions for which

$$\begin{cases} y < k_5, & \theta > 0 \\ y > k_6, & \theta < 0 \end{cases}$$

The power function for $\theta > 0$ is given as

$$P_0 = F\left(b + \frac{2\theta}{\sigma a \sqrt{\dfrac{2(n_1 + n_2)(1 + \rho)}{n_1 n_2}}}\right)$$

$$F(b) = \varepsilon$$

$$k_5 = \sigma_y b$$

The two pairs of symmetric quantiles that maximize P_0 are those of orders $p_1 = .2703$, $p_2 = .7297$.

For $n_1 = n_2 = n = 200$, $\varepsilon = .01$, and using the optimum quantiles, one obtains

$$P_0 = F\left(-2.326 + 9.0\,\frac{\theta}{\sigma}\right)$$

$$\sigma_y = 2.222\sigma\left(\frac{n_1 + n_2}{n_1 n_2}\right)^{1/2} = .222\sigma$$

$$k_5 = -5.169\sigma\left(\frac{n_1 + n_2}{n_1 n_2}\right)^{1/2} = -.5169\sigma$$

Hence, for $\theta > 0$, if

$$y = z(.7297) - z'(.7297) + z(.2703) - z'(.2703)$$

$$> -5.169\sigma\left(\frac{n_1 + n_2}{n_1 n_2}\right)^{1/2}$$

accept H_0. Otherwise reject H_0. For $\theta < 0$, if

$$y < 5.169\sigma\left(\frac{n_1 + n_2}{n_1 n_2}\right)^{1/2}$$

accept H_0. Otherwise reject H_0. Figure 8 shows the power and efficiency of the test using the two optimum pairs of quantiles.
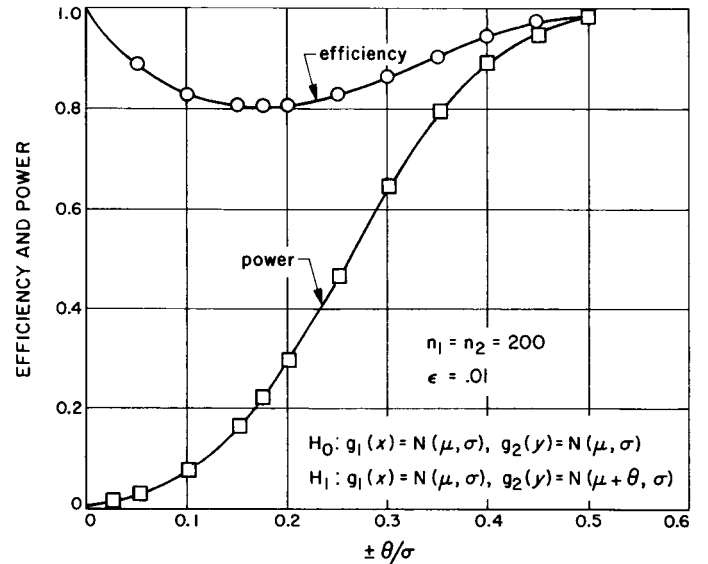
Fig. 8. Power and efficiency of Test D_2 using two pairs of quantiles

D. Test D_4: Four Pairs of Quantiles

Let z_i $(i = 1, 2, 3, 4)$ be the sample quantiles of the first population of orders p_i, such that $p_1 + p_4 = p_2 + p_3 = 1$, and let z_i' be the corresponding sample quantiles of the second population.

Form

$$w_i = z_i - z_i \qquad (i = 1, 2, 3, 4)$$

and

$$x_1 = w_1 + w_4$$

$$x_2 = w_2 + w_3$$

The test will be made on the statistic given by the linear combination

$$y = \alpha x_1 + \beta x_2$$

The parameters α and β, as well as the optimum orders of the quantiles, will be determined so as to maximize P_0. Omitting much of the details, one has, for $n_1 = n_2 = n$,

Under H_0:

$$E(y) = 0$$

$$\mathrm{Var}(y) = \frac{8\alpha\beta a_1 a_2 \sigma^2}{n} (\rho_{12} + \rho_{13})$$

$$+ \frac{4\sigma^2}{n} [\alpha^2 a_1^2 (1 + \rho_{14}) + \beta^2 a_2^2 (1 + \rho_{23})] = \sigma^2 \gamma^2$$

where

$$a_i^2 = \frac{F(\zeta_i)[1 - F(\zeta_i)]}{f^2(\zeta_i)} \qquad (i = 1, 2)$$

and ρ_{ij} denotes the correlation between z_i and z_j as well as that between z_i' and z_j'.

Under H_1:

$$E(y) = -2\theta(\alpha + \beta)$$

$$\mathrm{Var}(y) = \sigma^2 \gamma^2$$

The power function for $\theta > 0$ is given by

$$P_0 = F\left[b + \frac{2(\alpha + \beta)\theta}{\sigma\gamma} \right]$$

$$F(b) = \varepsilon$$

$$k = \sigma\gamma b$$

As in Test A_4, the orders of the four pairs of quantiles and weights α and β which maximize P_0 are

$$p_1 = .1068 \qquad p_2 = .3512$$

$$p_4 = .8932 \qquad p_3 = .6488$$

$$\alpha = .192 \qquad \beta = .308$$

Inserting these values and assuming $n = 200$, $\varepsilon = .01$, one has, finally,

$$P_0 = F\left(-2.326 + 9.586\frac{\theta}{\sigma} \right)$$
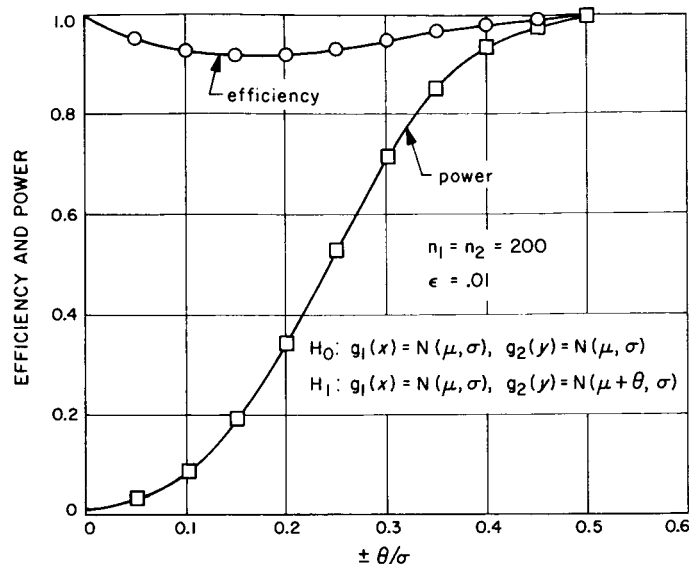
Fig. 9. Power and efficiency of Test D_4 using four pairs of quantiles

Figure 9 shows the power and efficiency of the test using four optimum quantiles. For this case

$$\sigma_y = \frac{1.4793\sigma}{n^{1/2}} = .1046\sigma$$

$$k = -\frac{3.441}{n^{1/2}} = .2433\sigma$$

so that for $\theta > 0$, if

$$y = .192\,[z(.1068) - z'(.1068)$$

$$+ z(.8932) - z'(.8932)]$$

$$+ .308\,[z(.3512) - z'(.3512)$$

$$+ z(.6488) - z'(.6488)] \qquad > -\frac{3.441}{n^{1/2}}$$

accept H_0. Otherwise reject H_0. For $\theta < 0$, if

$$y < \frac{3.441\sigma}{n^{1/2}}$$

accept H_0. Otherwise reject H_0.

In each case, since the critical region depends upon σ and *not upon* θ, the tests are seen to be uniformly most

powerful (among quantile tests). Figures 7 through 9 show that as $|\theta/\sigma|$ increases from zero, the efficiency of each test decreases from 1 to a minimum value and then increases and approaches 1 asymptotically. The minimum efficiency is about .62 using one pair of quantiles, about .80 using two pairs of quantiles, and about .91 using four pairs of quantiles. Thus the efficiency of this test, as in previous tests, is quite high.

E. Test E: Determination of P'_0

In Test E we are testing the null hypothesis

$$H_0: \quad g_1(x) = N(0, \sigma), \quad g_2(y) = N(0, \sigma)$$

against the alternative hypothesis

$$H_1: \quad g_1(x) = N(0, \sigma), \quad g_2(y) = N(0, \theta\sigma), \quad \theta > 0$$

where σ is unknown; we will assume $n_1 = n_2 = n = 200$. Since σ is not known, the distribution of any test statistic must be independent of σ. Under this restraint, the best statistic using all the sample values is given by

$$s = \frac{\sum\limits_{i=1}^{n} x_i^2}{\sum\limits_{i=1}^{n} y_i^2}$$

However, it is more convenient to use instead the equivalent statistic

$$r = \tfrac{1}{2}\ln s = \tfrac{1}{2}\ln\left(\frac{\dfrac{1}{n}\sum\limits_{i=1}^{n} x_i^2}{\dfrac{1}{n}\sum\limits_{i=1}^{n} y_i^2}\right)$$

Now, under H_0:

$$r = \tfrac{1}{2}\ln s = \tfrac{1}{2}\ln\left(\frac{\dfrac{1}{n}\sum\limits_{i=1}^{n} x_i^2/\sigma^2}{\dfrac{1}{n}\sum\limits_{i=1}^{n} y_i^2/\sigma^2}\right)$$

and s has the F-distribution with n degrees of freedom in the numerator and denominator. Since n is large, r is approximately normal (Ref. 7, pp. 378–379), with zero mean and variance equal to 1/n. This approximation will be used to compute P'_0.

Under H_1:

$$r = \tfrac{1}{2}\ln s = \tfrac{1}{2}\ln\left(\frac{\dfrac{1}{\theta^2 n}\sum\limits_{i=1}^{n} x_i^2/\sigma^2}{\dfrac{1}{n}\sum\limits_{i=1}^{n} y_i^2/\theta^2\sigma^2}\right)$$

$$= \tfrac{1}{2}\ln\left(\frac{\dfrac{1}{n}\sum\limits_{i=1}^{n} x_i^2/\sigma_2}{\dfrac{1}{n}\sum\limits_{i=1}^{n} y_i^2/\theta^2\sigma^2}\right) - \tfrac{1}{2}\ln\theta^2$$

Thus, under the alternative hypothesis, r is also approximately normal, with mean equal to $-\tfrac{1}{2}\ln\theta^2$ and variance equal to 1/n. The critical regions are easily determined to be the regions for which

$$\begin{cases} r < k_7, & \theta > 1 \\ r > k_8, & \theta < 1 \end{cases}$$

For $\varepsilon = .01$,

$$P'_0 = F(-2.326 + 7.07\ln\theta^2) \qquad \theta > 1, \quad k_7 = -\frac{2.326}{n^{1/2}}$$

$$P'_0 = 1 - F\left(2.326 - 7.07\ln\frac{1}{\theta^2}\right) \qquad \theta < 1, \quad k_8 = \frac{2.326}{n^{1/2}}$$

F. Test E_1: One Pair of Quantiles

The orders of the quantiles used in the quantile tests will be those which minimize the variance of the estimate of σ from a single set of samples. Since μ_1 and μ_2 are known, it is possible to obtain a consistent estimate of σ using one optimum quantile of order p = .058 or p = .942 as determined by J. Ogawa (Ref. 6). (If μ_1 and μ_2 are not known, it is still possible, however, to obtain a non-consistent test statistic by using the medians; but since the power is extremely poor as well as independent of n, this test will not be considered).

Thus let z be the sample quantile of order p = .942 of the samples taken from the first population, and let z' be the corresponding sample quantile from the second population. The test statistic that will be used to eliminate dependence on σ is given by $u = z/z'$. In order to specify a critical region for a given ε, it is necessary to determine the distribution of u. In general, if x and y are normal random variables and are distributed $N(m_1, \sigma_1)$ and

$N(m_2, \sigma_2)$, respectively, then $u = x/y$ is shown in Appendix A to have a density function given by

$$h(u) = \frac{\sigma_1\sigma_2 \exp - \dfrac{1}{2}\left(\dfrac{\sigma_1^2 m_2^2 + \sigma_2^2 m_1^2}{\sigma_1^2\sigma_2^2}\right)}{\pi(\sigma_2^2 u^2 + \sigma_1^2)} + \frac{\sigma_2^2 m_1 u + \sigma_1^2 m_2}{(2\pi)^{1/2}(\sigma_2^2 u^2 + \sigma_1^2)^{3/2}} \exp\left\{-\frac{1}{2}\left[\frac{(m_2 u - m_1)^2}{\sigma_2^2 u^2 + \sigma_1^2}\right]\right\}$$

$$\times \left[-1 + 2F\left(\frac{\sigma_2^2 m_1 u + \sigma_1^2 m_2}{\sigma_1\sigma_2(\sigma_2^2 u^2 + \sigma_1^2)^{1/2}}\right)\right] \qquad (-\infty < u < \infty)$$

Thus, since under H_0

$$E(z) = E(z') = \sigma\zeta$$

$$\text{Var}(z) = \text{Var}(z') = \sigma^2 a^2$$

where

$$a^2 = \frac{F(\zeta)[1 - F(\zeta)]}{nf^2(\zeta)}$$

whereas, under H_1,

$$E(z) = \sigma\zeta$$

$$E(z') = \sigma\theta\zeta$$

$$\text{Var}(z) = \sigma^2 a^2$$

$$\text{Var}(z') = \sigma^2\theta^2 a^2$$

one has, for p = .942,

$$h(u\,|\,H_0) = \frac{\exp(-121.679)}{\pi(u^2 - 1)} + \frac{4.4007(u+1)}{(u^2+1)^{3/2}} \exp\left\{-\left[\frac{60.481(u-1)^2}{u^2+1}\right]\right\} \times \left[-1 + 2F\left(\frac{11.031(u+1)}{(u^2+1)^{1/2}}\right)\right]$$

$$(-\infty < u < \infty)$$

$$h(u\,|\,H_1) = \frac{\theta\exp(-121.679)}{\pi(\theta^2 u^2 - 1)} + \frac{4.4007\theta(\theta u+1)}{(\theta^2 u^2+1)^{3/2}} \exp\left\{-\left[\frac{60.481(\theta u-1)^2}{\theta^2 u^2+1}\right]\right\} \times \left[-1 + 2F\left(\frac{11.031(\theta u+1)}{(\theta^2 u^2+1)^{1/2}}\right)\right]$$

Since

$$\int_{-\infty}^{\infty} \frac{\theta}{\pi(\theta^2 a^2 + 1)}\,du = 1$$

and

$$2F\left(\frac{11.031(\theta u + 1)}{(\theta^2 u^2 + 1)^{1/2}}\right) \cong 2$$

for all values of u and θ for which

$$\exp\left[\frac{-60.481(\theta u - 1)^2}{\theta^2 u^2 + 1}\right]$$

is not nearly equal to zero, $h(u|H_0)$ and $h(u|H_1)$ can be written as follows:

$$h(u|H_0) = \frac{4.4007(u+1)}{(u^2+1)^{3/2}} \exp\left[-\frac{60.841(u-1)^2}{u^2+1}\right]$$

$$h(u|H_1) = \frac{4.4007\theta(\theta u+1)}{(\theta^2 u^2+1)^{3/2}} \exp\left[-\frac{60.841(\theta u-1)^2}{\theta^2 u^2+1}\right]$$

Moreover, the mean and mode of $h(u|H_0)$ and $h(u|H_1)$ are approximately $u=1$ and $u=1/\theta$, respectively, so that for $\theta > 1$, $h(u|H_1)$ is shifted to the left of $h(u|H_0)$ and for $\theta < 1$, $h(u|H_1)$ is shifted to the right of $h(u|H_0)$. Hence, the critical regions will be taken as the regions for which

$$\begin{cases} u < k_9, & \theta > 1 \\ u > k_{10}, & \theta < 1 \end{cases}$$

where k_9 and k_{10} are defined by

$$\left.\begin{array}{l} \displaystyle\int_{-\infty}^{k_9} h(u|H_0)\,du = \varepsilon \\[3mm] \displaystyle\int_{-\infty}^{k_9} h(u|H_1)\,du = P_0 \end{array}\right\} \quad \theta > 1$$

$$\left.\begin{array}{l} \displaystyle\int_{-\infty}^{k_{10}} h(u|H_0)\,du = 1-\varepsilon \\[3mm] \displaystyle\int_{-\infty}^{k_{10}} h(u|H_1)\,du = 1-P_0 \end{array}\right\} \quad \theta < 1$$

The values of k_9 and k_{10} were determined to be, for $\varepsilon = .01$,

$$k_9 = .738$$

$$k_{10} = 1.354$$

Thus for $\theta > 1$, if

$$u = \frac{z(.9424)}{z'(.9424)} > .738$$

accept H_0. Otherwise reject H_0. For $\theta < 1$, if $u < 1.354$, accept H_0. Otherwise reject H_0. Figure 10 shows the efficiency and power of the one-pair-of-quantiles test as a function of θ.
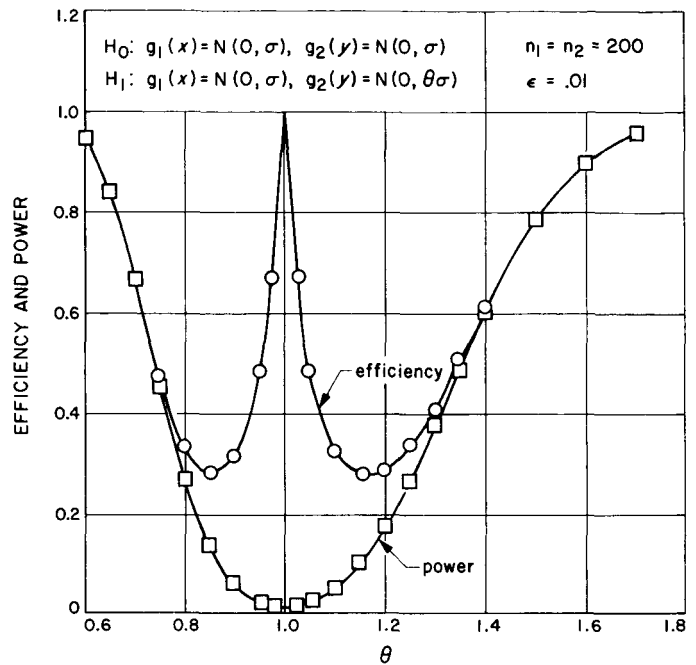
Fig. 10. Power and efficiency of Test E_1 using one pair of quantiles

G. Test E_2: Two Pairs of Quantiles

The orders of the optimum two pairs of symmetric quantiles that will be used in the test using two pairs of quantiles are

$$p_1 = .069 \qquad p_2 = .931$$

Hence, let z_1 and z_2 be the two sample quantiles taken from the first population of orders p_1 and p_2, and let z_1' and z_2' be the corresponding sample quantiles from the second population. Now let $x_1 = z_2 - z_1$ and $x_2 = z_2' - z_1'$

Under H_0:

$$E(x_1) = E(x_2) = 2\sigma\zeta_2$$

$$\text{Var}(x_1) = \text{Var}(x_2) = 2\sigma^2 a^2(1-\rho)$$

where

$$a^2 = \frac{F(\zeta_1)[1-F(\zeta_2)]}{nf^2(\zeta_1)}$$

and ρ is the correlation between z_1 and z_2.

Under H_1:

$$E(x_1) = 2\sigma\zeta_2$$

$$\text{Var}(x_1) = 2\sigma^2 a^2(1-\rho)$$

$$E(x_2) = 2\theta\sigma\zeta_2$$

$$\text{Var}(x_2) = 2\sigma^2\theta^2 a^2(1-\rho)$$

Now using $u = x_1/x_2$ as the test statistic, one has, for $p_1 = .069$, $p_2 = .931$,

$$h(u|H_0) = \frac{\exp(-260.76)}{\pi(u^2+1)} + \frac{6.442(u+1)}{(u^2+1)^{3/2}} \exp\left\{-\left[\frac{136.38(u-1)^2}{u^2+1}\right]\right\} \times \left[-1 + 2F\left(\frac{16.14(u+1)}{u^2+1}\right)\right]$$

$$h(u|H_1) = \frac{\theta \exp(-260.76)}{\pi(\theta^2u^2+1)} + \frac{6.442\theta(\theta u+1)}{(\theta^2u^2+1)^{3/2}} \exp\left\{-\left[\frac{136.38(\theta u-1)^2}{\theta^2u^2+1}\right]\right\} \times \left[-1 + 2F\left(\frac{16.14(\theta u+1)}{\theta^2u^2+1}\right)\right]$$

It is easily seen again, in the same sense as before, that one can express $h(u|H_0)$ and $h(u|H_1)$ as

$$h(u|H_0) = \frac{6.442(u+1)}{(u^2+1)^{3/2}} \exp\left[-\frac{136.38(u-1)^2}{u^2+1}\right]$$

$$h(u|H_1) = \frac{6.442\theta(\theta u+1)}{(\theta^2u^2+1)^{3/2}} \exp\left[-\frac{136.38(\theta u-1)^2}{\theta^2u^2+1}\right]$$

Taking the critical regions as

$$\begin{cases} u < k_{11}, & \theta > 1 \\ u > k_{12}, & \theta < 1 \end{cases}$$

k_{11} and k_{12} were found to be, for $\varepsilon = .01$,

$$k_{11} = .814 \qquad k_{12} = 1.228$$

Thus, for $\theta > 1$, if

$$u = \frac{x_1}{x_2} = \frac{z(.931) - z(.069)}{z'(.931) - z'(.069)} > .814$$

accept H_0. Otherwise reject H_0. For $\theta < 1$, if $u < 1.228$ accept H_0. Otherwise reject H_0. Figure 11 shows the efficiency and power of the test using the best two pairs of symmetric quantiles.

H. Test E_i: Four Pairs of Quantiles

The orders of the four pairs of symmetric quantiles to be used in the four-quantile test are

$$p_1 = .023 \qquad p_2 = .127$$

$$p_4 = .977 \qquad p_3 = .873$$
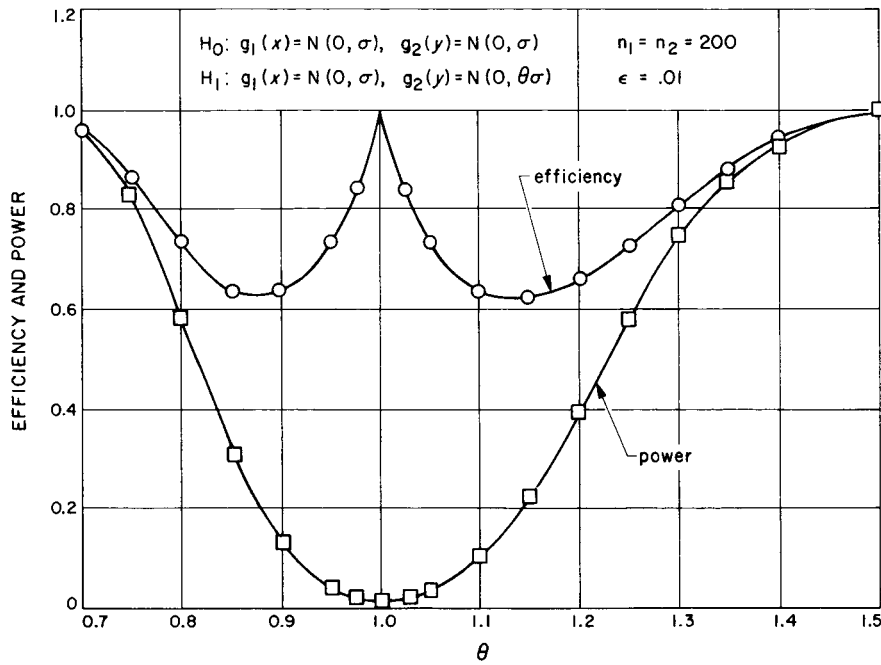
Fig. 11. Power and efficiency of Test E_2 using two pairs of quantiles

Hence, let z_i ($i = 1, 2, 3, 4$) be four sample quantiles from the first population with orders p_i, and let z_i' be the corresponding sample quantiles from the second population. Furthermore, define

$$x_1 = \alpha (z_4 - z_1) \qquad x_1' = \alpha (z_4' - z_1')$$

$$x_2 = \beta (z_3 - z_2) \qquad x_2' = \beta (z_3' - z_2')$$

and

$$y_1 = x_1 + x_2 \qquad y_2 = x_1' + x_2'$$

With weights $\alpha = .116$, $\beta = .236$, and the given orders of the quantiles, y_1 and y_2 are the best estimators of the standard deviation of the first and second population, respectively, each using two pairs of symmetric quantiles.

Under H_0:

$$E(y_1) = E(y_2) = 2\sigma (\alpha \zeta_4 + \beta \zeta_3)$$

$$Var(y_1) = Var(y_2) = 2\sigma^2 [\alpha^2 a_1^2 (1 - \rho_{14})$$
$$+ \beta^2 a_2^2 (1 - \rho_{23}) + 2\alpha\beta a_1 a_2 (\rho_{12} - \rho_{13})] = 2\sigma^2 \gamma^2$$

where

$$a_i^2 = \frac{F(\zeta_i)[1 - F(\zeta_i)]}{nf^2(\zeta_i)} \qquad (i = 1, 2)$$

and ρ_{ij} = the correlation between z_i and z_j

Under H_1:

$$E(y_1) = 2\sigma (\alpha \zeta_4 + \beta \zeta_3)$$

$$Var(y_1) = 2\sigma^2 \gamma^2$$

$$E(y_2) = 2\sigma\theta (\alpha \zeta_4 + \beta \zeta_3)$$

$$Var(y_2) = 2\theta^2 \sigma^2 \gamma^2$$

Using $u = y_1/y_2$ as the test statistic, one uses for the distribution of u, as before

$$h(u \mid H_0) = \frac{7.257(u + 1)}{(u^2 + 1)^{3/2}} \exp \left[-\frac{165.445(u - 1)^2}{u^2 + 1} \right]$$

$$h(u \mid H_1) = \frac{7.257\theta (\theta u + 1)}{(\theta^2 u^2 + 1)^{3/2}} \exp \left[-\frac{165.445(\theta u - 1)^2}{\theta^2 u^2 + 1} \right]$$

Taking the critical regions as

$$\begin{cases} u < k_{13}, & \theta > 1 \\ u > k_{14}, & \theta < 1 \end{cases}$$

k_{13} and k_{14} are found to be, for $\epsilon = .01$:

$$k_{13} = .834 \qquad k_{14} = 1.200$$

Thus, for $\theta > 1$, if

$$\mu = \frac{.116\,[z\,(.977) - z\,(.023)] + .236\,[z\,(.873) - z\,(.127)]}{.116\,[z'\,(.977) - z'\,(.023)] + .236\,[z'\,(.873) - z'\,(.127)]}$$

$$> .834$$

accept H_0. Otherwise reject H_0. For $\theta < 1$, if $\mu < 1.200$, accept H_0. Otherwise reject H_0. Figure 12 shows the efficiency and power of the test using the best four pairs of symmetric quantiles.

It can be seen from Fig. 10 through 12 that as θ increases or decreases from $\theta = 1$, the efficiency decreases from 1 to a minimum and then increases and approaches 1 asymptotically. The minimum efficiency is about .28 using one pair of quantiles, about .62 using two pairs of quantiles, and about .81 using four pairs of quantiles.

To convert test E to Test E', if z and z' are sample quantiles of order p taken from the first and second populations, respectively, under Test E', then $z - \mu_1$ and $z' - \mu_2$ should be used as the corresponding sample quantiles under Test E.
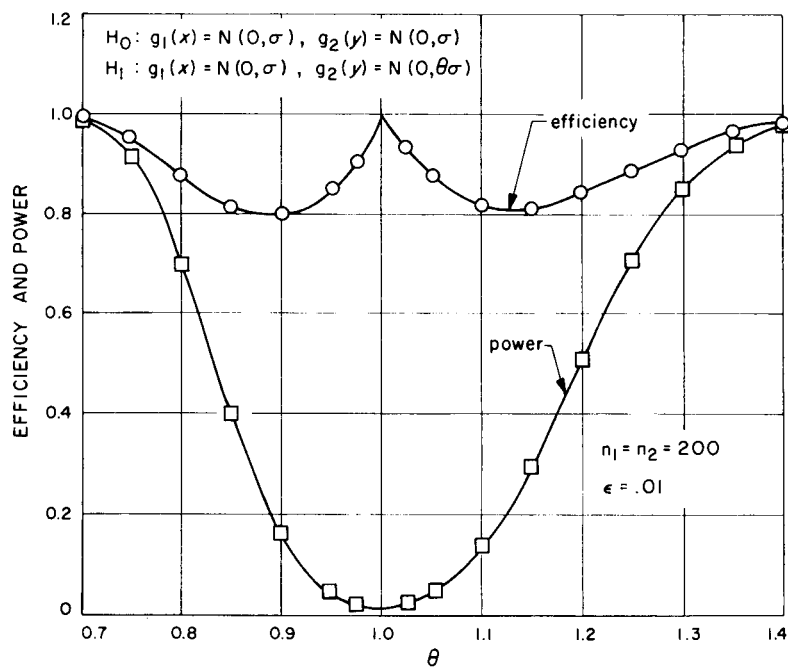
Fig. 12. Power and efficiency of Test E₄ using four pairs of quantiles

VII. TEST F: TESTS OF INDEPENDENCE AND ESTIMATION OF THE CORRELATION COEFFICIENT ρ

A. Statement of the Problem

Given a set of n independent pairs of observations $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ taken from two normally distributed populations with known means and variances, one is often interested in the answers to the following two questions:

1. Can we assert that the set of observations

$$x = \{x_1, x_2, \cdots, x_n\}$$

   is independent of the set of observations

$$y = \{y_1, y_2, \cdots, y_n\}?$$

2. What can be said about the correlation between them, if any?

To answer the first question, the problem of testing the null hypothesis

$$H_0: \quad g_1(x) = N(0,1), \quad g_2(y) = N(0,1), \quad \rho = 0$$

against the alternative hypothesis

$$H_1: \quad g_2(x) = N(0,1), \quad g_2(y) = N(0,1), \quad \rho \neq 0$$

will be considered. (Because of the assumption of known means and variances, we can, without loss of generality, assume standard normal distributions.)

To answer the second question, unbiased estimators of ρ will be constructed. The power functions of the tests will be derived, and the efficiencies of the tests will be determined relative to the best test using the entire sample. The efficiencies of the estimators will also be determined relative to the sample correlation coefficient for the case $\rho = 0$. One, two, and four pairs of quantiles will be used for the tests and estimators; n is assumed to be large (≥ 200). An application will be given to telemetering micrometeoroid measurements from spacecraft.

B. Test F_1: One Pair of Quantiles

It is necessary, at this point, to form two new sets of values $\{u_i\}$ and $\{v_i\}$ from the sample values $\{x_i\}$ and $\{y_i\}$ by means of the linear transformations

$$u_i = \frac{2^{1/2}}{2}(x_i + y_i)$$

$$v_i = \frac{2^{1/2}}{2}(-x_i + y_i)$$

It is easily verified that, under H_0

$$E(u_i) = E(v_i) = 0$$

$$\text{Var}(u_i) = \text{Var}(v_i) = 1$$

$$E(u_i v_i) = 0$$

and under H_1

$$E(u_i) = E(v_i) = 0$$

$$\text{Var}(u_i) = 1 + \rho$$

$$\text{Var}(v_i) = 1 - \rho$$

$$E(u_i v_i) = 0$$

so that the set of values $\{u_i\}$ is independent of the set of values $\{v_i\}$ *under both hypotheses*. All the tests and estimators will be based on the quantiles of the transformed sets of variables $\{u_i\}$ and $\{v_i\}$, which are all normally distributed.

Now let z and z' denote the quantiles of order p of $\{u_i\}$ and $\{v_i\}$, respectively. The test will be made on the statistic $w = z - z'$.

Under H_0:

$$E(z) = E(z') = \zeta$$

$$\text{Var}(z) = \text{Var}(z') = a^2$$

$$E(w) = 0$$

$$\text{Var}(w) = 2a^2$$

where

$$a^2 = \frac{F(\zeta)[1 - F(\zeta)]}{nf^2(\zeta)}$$

Under H_1:

$$E(x) = \zeta(1 + \rho)^{1/2}$$

$$\text{Var}(z) = a^2(1 + \rho)$$

$$E(z') = \zeta(1 - \rho)^{1/2}$$

$$\text{Var}(z') = a^2(1 - \rho)$$

$$E(w) = \zeta[(1 + \rho)^{1/2} - (1 - \rho)^{1/2}]$$

$$\text{Var}(w) = a^2(1 + \rho) + a^2(1 - \rho) = 2a^2$$

The best critical region is determined by the likelihood ratio inequality

$$\frac{L(w|H_0)}{L(w|H_1)}$$

$$= \frac{\exp\left\{-\frac{1}{2}\left[\frac{w^2}{2a^2}\right]\right\}}{\exp\left\{-\frac{1}{2}\left[\frac{(w - \zeta[(1+\rho)^{1/2} - (1-\rho)^{1/2}]^2}{2a^2}\right]\right\}} < c$$

which reduces to

$$\begin{cases} w > k_1 & \text{for } \rho > 0 \\ w < k_2 & \text{for } \rho < 0 \end{cases}$$

as the regions providing the maximum power.

To determine P_0, one has, for $\rho > 0$,

Under H_0:

$$\Pr(w < k_1) = F\left(\frac{k_1}{a(2)^{1/2}}\right)$$

$$= F(b_1) = 1 - \varepsilon, \quad k_1 = ab_1(2)^{1/2}$$

and, under H_1:

$$\Pr(w < k_1) = F\left[\frac{k_1 - \zeta[(1+\rho)^{1/2} - (1-\rho)^{1/2}]}{a(2)^{1/2}}\right]$$

$$= F\left[b_1 - \frac{\zeta[(1+\rho)^{1/2} - (1-\rho)^{1/2}]}{a(2)^{1/2}}\right] = 1 - P_0$$

(12)

P_0 will be a maximum if the order of the quantiles z and z' is chosen to maximize ζ/a. From Test B_1 in Part IV,

we know that this maximum occurs when $p = F(1.575) = .9424$. For this value of p, Eq. 12 becomes

$$P_0 = 1 - F[b_1 - .5515(n)^{1/2}][(1 + \rho)^{1/2} - (1 - \rho)^{1/2}]$$

Using all the transformed values $\{u_i\}$ and $\{v_i\}$, the likelihood ratio inequality gives, as the best critical region,

$$w' = \sum_{i=1}^{n}\left[(\rho - 1)\mu_i^2 + (\rho + 1)v_i^2\right]\begin{cases} < k_3 & \text{for } \rho > 0 \\ > k_4 & \text{for } \rho < 0 \end{cases}$$

Thus w' is asymptotically normally distributed and, under H_0

$$E(w') = 2n\rho$$

$$\text{Var}(w') = 4n(\rho^2 + 1)$$

while under H_1

$$E(w') = 0$$

$$\text{Var}(w') = 4n(1 - \rho^2)^2$$

Hence, assuming the limiting distribution of w' (which is a very good approximation for $n = 200$) one has

Under H_0, for $\rho > 0$,

$$\Pr(w' < k_3) = F\left(\frac{k_3 - 2n\rho}{2(n[\rho^2 + 1])^{1/2}}\right) = F(b_2) = \varepsilon$$

$$k_3 = 2b_2(n[\rho^2 + 1])^{1/2} + 2n\rho$$

Under H_1,

$$\Pr(w' < k_3) = F\left(\frac{k_3}{2(1 - \rho^2)(n)^{1/2}}\right)$$

$$= F\left[\frac{1}{1 - \rho^2}\left(b_2(\rho^2 + 1)^{1/2} + \rho(n)^{1/2}\right)\right] = P_0'$$

Figure 13 shows, for $n = 200$, $\varepsilon = .01$, the power and efficiency of Test F_1 using one pair of optimum quantiles. The efficiency is never less than .30, approaches 1 as $\rho \to 0$, and approaches a number close to, but not equal to, 1 as $|\rho| \to 1$. Under these conditions $b_1 = 2.326$, $b_2 = -2.326$, $k_1 = 6.6424/(n)^{1/2} = .4697$.
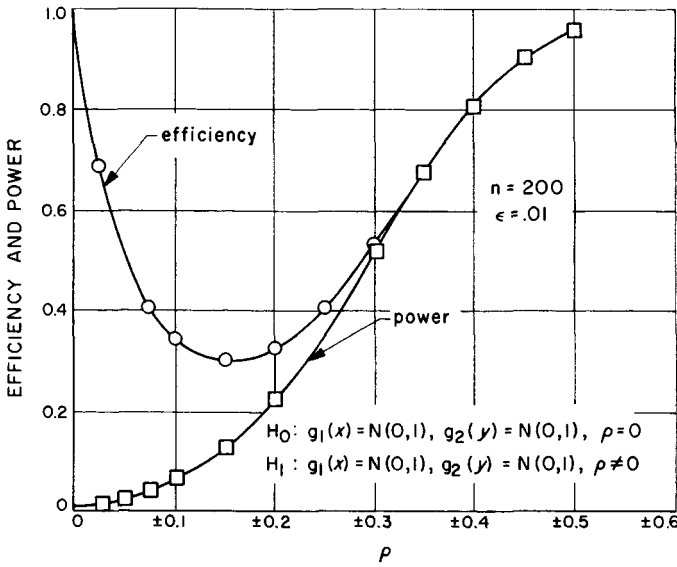
Fig. 13. Power and efficiency of Test F_1 using one pair of quantiles

Thus, for $\rho > 0$, if

$$w = z\,(.9424) - z'\,(0.9424) < \frac{6.6424}{(n)^{1/2}}$$

accept H_0. Otherwise reject H_0.

If $\rho < 0$, accept H_0 if $w > -6.6424/(n)^{1/2}$. Otherwise reject H_0.

C. Test F_2: Two Pairs of Quantiles

Let z_1 and z_2 be the quantiles of $\{u_i\}$ of orders p_1 and $p_2 = 1 - p_1$, and let z_1' and z_2' be the corresponding quantiles of $\{v_i\}$. Furthermore, let $w_1 = z_1 - z_1'$, $w_2 = z_2 - z_2'$. The test will be based on the values of w_1 and w_2.

Under H_0:

$$E\,(z_1) = E\,(z_1') = -\zeta_2$$

$$E\,(z_2) = E\,(z_2') = \zeta_2$$

$$E\,(w_1) = E\,(w_2) = 0$$

$$Var\,(z_1) = Var\,(z_1') = Var\,(z_2) = Var\,(z_2')$$

$$= \frac{F\,(\zeta_2)\,[1 - F\,(\zeta_2)]}{nf^2\,(\zeta_2)} = a^2$$

Hence,

$$E\,(w_1) = E\,(w_2) = 0$$

$$Var\,(w_1) = Var\,(w_2) = 2a^2$$

Under H_1:

$$E\,(z_1) = -\zeta_2\,(1 + \rho)^{1/2}$$

$$E\,(z_1') = -\zeta_2\,(1 - \rho)^{1/2}$$

$$E\,(z_2) = \zeta_2\,(1 + \rho)^{1/2}$$

$$E\,(z_2') = \zeta_2\,(1 - \rho)^{1/2}$$

$$E\,(w_1) = -\zeta_2\,[(1 + \rho)^{1/2} - (1 - \rho)^{1/2}]$$

$$E\,(w_2) = \zeta_2\,[(1 + \rho)^{1/2} - (1 - \rho)^{1/2}]$$

$$Var\,(z_1) = Var\,(z_2) = (1 + \rho)a^2$$

$$Var\,(z_1') = Var\,(z_2') = (1 - \rho)a^2$$

$$Var\,(w_1) = Var\,(w_2) = 2a^2$$

Simplifying the likelihood ratio inequality results in the best critical regions

$$s = (w_2 - w_1)\begin{cases} > k_5 \text{ for } \rho > 0 \\ < k_6 \text{ for } \rho < 0 \end{cases}$$

Under H_0:

$$E\,(s) = 0$$

$$Var\,(s) = 4a^2\,(1 - \rho_{12})$$

where ρ_{12} denotes the correlation between z_1 and z_2, as well as the correlation between z_1' and z_2'. For $\rho > 0$

$$Pr\,(s < k_5) = F\left[\frac{k_5}{2a\,(1 - \rho_{12})^{1/2}}\right] = F\,(b_3)$$

$$= 1 - \varepsilon, \quad k_5 = 2ab_3\,(1 - \rho_{12})^{1/2}$$

Under H_1:

$$E\,(s) = 2\zeta_2\,[(1 + \rho)^{1/2} - (1 - \rho)^{1/2}]$$

$$Var\,(s) = 4a^2\,(1 - \rho_{12})$$

$$Pr\,(s < k_5) = F\left[\frac{k_5 - 2\zeta_2\,[(1 + \rho)^{1/2} - (1 - \rho)^{1/2}]}{2a\,(1 - \rho_{12})^{1/2}}\right]$$

$$= F\left[b_3 - \frac{\zeta_2\,[(1 + \rho)^{1/2} - (1 - \rho)^{1/2}]}{a\,(1 - \rho_{12})^{1/2}}\right]$$

$$= 1 - P_0$$

The quantity to be maximized in this test, in order to maximize P_0, is $\zeta_2/a\,(1 - \rho_{12})^{1/2}$. It was found in Test B_2 that this maximum occurs at $p_2 = .9310$. Hence, for this value of p_2 the optimum power function is given by

$$P_0 = 1 - F\,\{b_3 - .8076\,(n)^{1/2}\,[(1 + \rho)^{1/2} - (1 - \rho)^{1/2}]\}$$
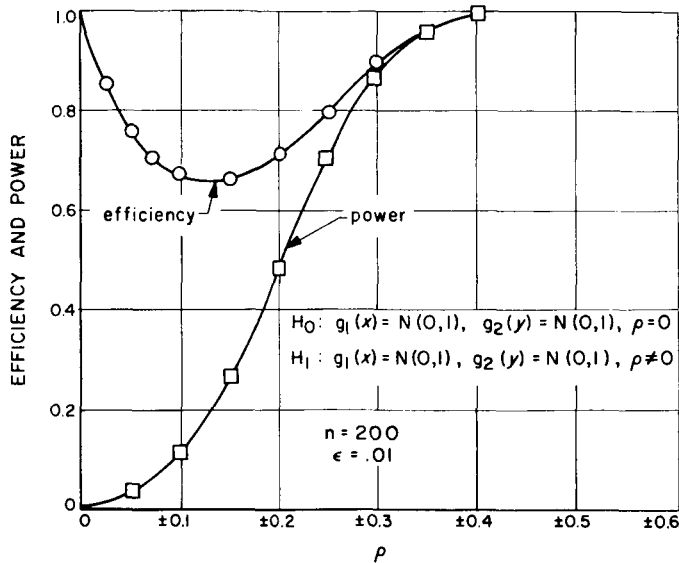
Fig. 14. Power and efficiency of Test F_2 using two pairs of quantiles

Figure 14 shows, for $n = 200$, $\epsilon = .01$, the power and efficiency of the test using two pairs of optimum symmetric quantiles. For this case, $b_3 = 2.326$ and $k_5 = 8.5432/(n)^{1/2} = .6041$

Thus for $\rho > 0$ if

$$s = z(.9310) - z'(.9310) - z(.0690)$$

$$+ z'(.0690) < \frac{8.5432}{n^{1/2}}$$

accept H_0. Otherwise reject H_0. For $\rho < 0$, if

$$s > \frac{-8.5432}{n^{1/2}},$$

accept H_0. Otherwise reject H_0. The efficiency never drops below .66.

D. Test F_4: Four Pairs of Quantiles

For this case, let z_i and z'_i ($i = 1, 2, 3, 4$) be four quantiles of $\{u_i\}$ and $\{v_i\}$, respectively, such that

$$p_1 + p_4 = p_2 + p_3 = 1$$

Furthermore, let $w_i = z_i - z'_i$, $s_1 = w_4 - w_1$, and

$$s_2 = w_3 - w_2.$$

The test will be made on the linear combination

$$t = \alpha s_1 + \beta s_2$$

and α and β, as well as the orders of the quantiles will be determined so as to maximize P_0.

Under H_0:

$$E(s_1) = E(s_2) = 0$$

$$\mathrm{Var}(s_1) = 4a_4^2(1 - \rho_{14})$$

$$\mathrm{Var}(s_2) = 4a_3^2(1 - \rho_{23})$$

$$E(t) = 0$$

$$\mathrm{Var}(t) = 4[\alpha^2 a_4^2(1 - \rho_{14}) + \beta^2 a_3^2(1 - \rho_{23})$$

$$+ 2\alpha\beta a_3 a_4(\rho_{12} - \rho_{13})] = 4\gamma^2$$

where

$$a_i^2 = \frac{F(\zeta_i)[1 - F(\zeta_i)]}{nf^2(\zeta_i)} \qquad (i = 3, 4)$$

and ρ_{ij} denotes the correlation between z_i and z_j, as well as the correlation between z'_i and z'_j.

Under H_1:

$$E(s_1) = 2\zeta_4[(1 + \rho)^{1/2} - (1 - \rho)^{1/2}]$$

$$\mathrm{Var}(s_1) = 4a_4^2(1 - \rho_{14})$$

$$E(s_2) = 2\zeta_3[(1 + \rho)^{1/2} - (1 - \rho)^{1/2}]$$

$$\mathrm{Var}(s_2) = 4a_3^2(1 - \rho_{23})$$

$$E(t) = 2[(1 + \rho)^{1/2} - (1 - \rho)^{1/2}](\alpha\zeta_4 + \beta\zeta_3)$$

$$\mathrm{Var}(t) = 4\gamma^2$$

Omitting the details, one obtains as the critical regions

$$t \begin{cases} > k_7 \text{ for } \rho > 0 \\ < k_8 \text{ for } \rho < 0 \end{cases}$$

The power function, for $\rho > 0$, is given by

$$P_0 = 1 - F\left[b_4 - \frac{[(1 + \rho)^{1/2} - (1 - \rho)^{1/2}](\alpha\zeta_4 + \beta\zeta_3)}{\gamma}\right]$$

where

$$F(b_4) = 1 - \epsilon \quad \text{and} \quad k_7 = 2\gamma b_4$$

We now know that for

$$\alpha = .116 \qquad \beta = .236$$
$$p_1 = .0230 \qquad p_2 = .1269$$
$$p_4 = .9770 \qquad p_3 = .8731$$

the quantity $\alpha \zeta_4 + \beta \zeta_3 / \gamma$ (and hence also P_0) will be a maximum. Thus the optimum power function is given by

$$P_0 = 1 - F \{ b_4 - .9080 (n)^{\frac{1}{2}} [(1 + \rho)^{\frac{1}{2}} - (1 - \rho)^{\frac{1}{2}}] \}$$

Figure 15 shows, for $n = 200$, $\varepsilon = .01$, the power and efficiency of the test using four pairs of optimum symmetric quantiles. For this case, $k_7 = .1814$.

Hence, for $\rho > 0$, if

$$t = .116 \, [z(.9770) - z'(.9770) - z(.0230) + z'(.0230)]$$

$$+ \, .236 \, [z(.8731) - z'(.8731) - z(.1269)$$

$$+ \, z'(.1269)] < \frac{2.5654}{n^{\frac{1}{2}}}$$

accept H_0. Otherwise reject H_0. For $\rho < 0$, if

$$t > \frac{-2.5654}{n^{\frac{1}{2}}},$$

accept H_0. Otherwise reject H_0.
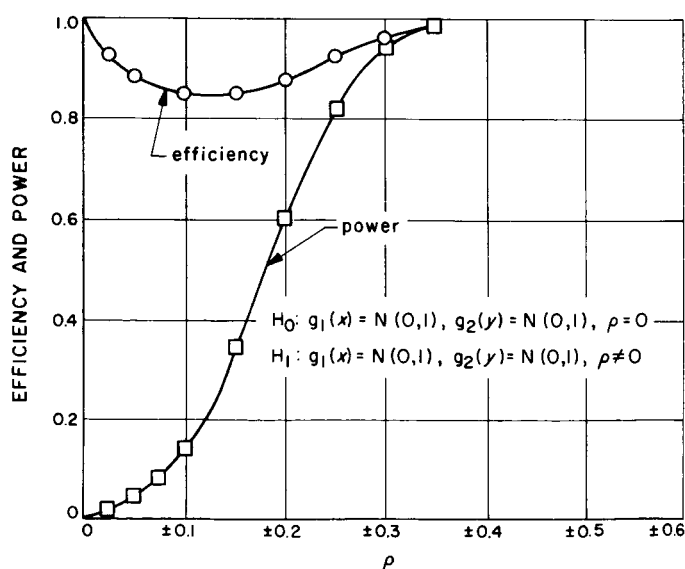
Fig. 15. Power and efficiency of Test F_4 using four pairs of quantiles

E. An Unbiased Estimator of ρ Using One Pair of Quantiles

With respect to the set of pairs of sample values $(x_1, y_1), \cdots, (x_n, y_n)$, the sets $\{x_i\}$ and $\{y_i\}$ are sample values of the random variables x and y, with a joint distribution given by

$$g_3(x, y) = \frac{1}{2\pi (1 - \rho^2)^{\frac{1}{2}}} \exp \left\{ - \frac{1}{2(1 - \rho^2)} \right.$$

$$\times \left. [x^2 - 2\rho xy + y^2] \right\}$$

The problem considered here is that of estimating ρ. By means of the linear transformation given above, a new set of sample values, $(u_1, v_1), \cdots, (u_n, v_n)$ is generated for which sets $\{u_i\}$ and $\{v_i\}$ can be considered as sample values of the random variables u and v with marginal distributions

$$h_1(u) = \frac{1}{[2\pi (1 + \rho)]^{\frac{1}{2}}} e^{-u^2/2(1+\rho)}$$

$$h_2(v) = \frac{1}{[2\pi (1 - \rho)]^{\frac{1}{2}}} e^{-v^2/2(1-\rho)}$$

and joint distribution

$$g_4(u, v) = h_1(u) h_2(v)$$

Unbiased estimators of ρ will be constructed using quantiles of $\{u_i\}$ and $\{v_i\}$, and the efficiencies of these estimators will be determined relative to the sample correlation coefficient r, the minimum-variance unbiased estimator of ρ, given by

$$r = \frac{\sum\limits_{i=1}^{n} (u_i - \bar{u})(v_i - \bar{v})}{\left\{ \sum\limits_{i=1}^{n} (u_i - \bar{u})^2 \sum\limits_{i=1}^{n} (v_i - \bar{v})^2 \right\}^{\frac{1}{2}}}$$

where

$$\bar{u} = \frac{1}{n} \sum_{i=1}^{n} u_i$$

$$\bar{v} = \frac{1}{n} \sum_{i=1}^{n} v_i$$

for the special case $\rho = 0$. Since the asymptotic variance, $\text{Var}\,(r|\rho = 0)$ is $1/n - 1$ (Ref. 8), the efficiency will be defined as

$$\text{Eff}\,(\hat{\rho}) = \frac{\text{Var}\,(r|\rho = 0)}{\text{Var}\,(\hat{\rho}|\rho = 0)} = \frac{1}{(n-1)\,\text{Var}\,(\hat{\rho}|\rho = 0)}$$

Hence, let z and z' denote the quantiles of order p of the $\{u_i\}$ and $\{v_i\}$, respectively. Then an unbiased estimator of ρ in terms of z and z' is given by

$$\hat{\rho} = \frac{z^2 - (z')^2}{2\,(a^2 + \zeta^2)}\,,$$

where

$$a^2 = \frac{F\,(\zeta)\,[1 - F\,(\zeta)]}{nf^2\,(\zeta)}$$

Since

$$E\,(z) = \zeta\,(1 + \rho)^{1/2}$$

$$E\,(z') = \zeta\,(1 - \rho)^{1/2}$$

$$\text{Var}\,(z) = a^2\,(1 + \rho)$$

$$\text{Var}\,(z') = a^2\,(1 - \rho)$$

one has, first of all,

$$E\,(\hat{\rho}) = \frac{1}{2\,(a^2 + \zeta^2)}\,[a^2\,(1 + \rho) - a^2\,(1 - \rho) + \zeta^2\,(1 + \rho) - \zeta^2\,(1 - \rho)] = \rho$$

so that $\hat{\rho}$ is seen to be unbiased. Also, noting the fact that if x is distributed $N\,(m, \sigma)$, then

$$\text{Var}\,(x^2) = 2\sigma^4 + 4m^2\sigma^2$$

and the variance of $\hat{\rho}$ is given by

$$\text{Var}\,(\hat{\rho}) = \frac{1}{4\,(a^2 + \zeta^2)^2}\,[\text{Var}\,(z^2) + \text{Var}\,(z')^2] = \frac{1}{4\,(a^2 + \zeta^2)^2}\,[2a^4\,(1 + \rho)^2 + 4\zeta^2 a^2\,(1 + \rho)^2$$

$$+ 2a^4\,(1 - \rho)^2 + 4\zeta^2 a^2\,(1 - \rho)^2] = \frac{a^2\,(a^2 + 2\zeta^2)\,(1 + \rho^2)}{(a^2 + \zeta^2)^2} \tag{13}$$

For $\rho = 0$, Eq. 13 can be written as

$$\text{Var}\,(\hat{\rho}|\rho = 0) = \frac{a^4 + 2\zeta^2 a^2}{a^4 + 2a^2\zeta^2 + \zeta^4} \cong \frac{1}{1 + \dfrac{\zeta^2}{2a^2}} \tag{14}$$

if one neglects the a^4 term in the numerator and denominator of Eq. 14 (this term is small compared to $\zeta^2 a^2$ and ζ^4 for large n). Now the approximate value of

$$\text{Var}\,(\hat{\rho}|\rho = 0)$$

in Eq. 14 is minimized if one chooses the order of z and z' to maximize ζ^2/a^2. It was found previously that $p = .9424$ will maximize ζ/a and will, of course, also maximize ζ^2/a^2. Moreover, since the curve defined by $\text{Var}(\hat{\rho}|\rho = 0)$ is quite flat around its minimum, the error involved in using $p = .9424$ instead of the true value of p which minimizes $\text{Var}(\hat{\rho}|\rho = 0)$ is small. Thus, using $p = .9424$, one obtains

$$\hat{\rho} = \frac{z^2 - (z')^2}{\dfrac{8.1552}{n} + 4.9612}$$

$$\text{Var}(\hat{\rho}|\rho = 0) = \frac{\dfrac{16.627}{n^2} + \dfrac{20.230}{n}}{\dfrac{16.627}{n^2} + \dfrac{20.230}{n} + 6.1535} \cong \frac{1}{1 + .3042n}$$

For $n = 200$,

$$\hat{\rho} = .200 \left[z^2 - (z')^2 \right]$$

$$\text{Var}(\hat{\rho}|\rho = 0) = .01624$$

$$\frac{1}{1 + .3042n} = .01617$$

$$\text{Eff}(\hat{\rho}) = .3094$$

F. An Unbiased Estimator of ρ Using Two Pairs of Quantiles

Let z_1, z_1', z_2, z_2' be defined as in Part VII, paragraph C. Then an unbiased estimator of ρ using two pairs of symmetric transformed sample quantiles is given by

$$\hat{\rho} = \frac{z_2^2 - (z_2')^2 + z_1^2 - (z_1')^2}{4(a^2 + \zeta_2^2)}$$

where

$$a^2 = \frac{F(\zeta_2)\left[1 - F(\zeta_2)\right]}{nf^2(\zeta_2)}$$

Then

$$E(\hat{\rho}) = \frac{1}{4(a^2 + \zeta_2^2)}\left[2\zeta_2^2(1 + \rho) + 2a^2(1 + \rho) - 2\zeta_2^2(1 - \rho) - 2a^2(1 - \rho)\right] = \rho$$

Noting that if x and y are distributed $N(m_1, \sigma_1)$ and $N(m_2, \sigma_2)$, respectively (Appendix B), then

$$E(x^2y^2) = m_1^2m_2^2 + \sigma_1^2m_2^2 + \sigma_2^2m_1^2 + 4\rho_{xy}\sigma_1\sigma_2m_1m_2 + \sigma_1^2\sigma_2^2(1 + \rho_{xy}^2)$$

and one has

$$\text{Var}(\hat{\rho}) = \frac{1}{16(a^2 + \zeta_2^2)^2}\left[\text{Var}(z_2^2 + z_1^2) + \text{Var}((z_2') + (z_1')^2)\right]$$

$$= \frac{1}{16(a^2 + \zeta_2^2)^2}\left[4a^4(1 + \rho_{12})^2 + 8a^2\zeta_2^2(1 - \rho_{12})\right]\left[(1 + \rho)^2 + (1 - \rho)^2\right]$$

$$= \frac{a^2\left[a^2(1 + \rho_{12}^2) + 2\zeta_2(1 - \rho_{12})\right](1 + \rho^2)}{2(a^2 + \zeta_2^2)^2} \tag{15}$$

where ρ_{12} denotes the correlation between z_1 and z_2.

For $\rho = 0$, Eq. 15 can be written as

$$\text{Var}\,(\hat{\rho}|\rho = 0) = \frac{a^4\,(1 + \rho_{12}^2) + 2a^2\zeta_2^2\,(1 - \rho_{12})}{2a^4 + 4a^2\zeta_2^2 + 2\zeta_2^4} \cong \frac{1}{2 + \zeta_2^2/a^2\,(1 - \rho_{12})} \tag{16}$$

neglecting the a^4 term in the numerator and denominator of Eq. 16 and taking $1 - \rho_{12} \cong 1$. The approximate value of Var $(\hat{\rho}|\rho = 0)$ in Eq. 16 is minimized by maximizing $\zeta^2/a^2\,(1 - \rho_{12})$. This maximum occurs, as found previously, at $p_2 = .9310$. Thus, using the values $p_1 = .0690$ and $p_2 = .9310$,

$$\hat{\rho} = \frac{z_2^2 - (z_2')^2 + z_1^2 + (z_1')^2}{\dfrac{14.5694}{n} + 8.7972}$$

$$\text{Var}\,(\hat{\rho}|\rho = 0) = \frac{\dfrac{13.3397}{n^2} + \dfrac{14.8330}{n}}{\dfrac{26.5334}{n^2} + \dfrac{32.042}{n} + 9.6738} \cong \frac{1}{2 + .6522n}$$

For $n = 200$

$$\hat{\rho} = .1127\,[z_2^2 - (z_2')^2 + z_1^2 - (z_1')^2]$$

$$\text{Var}\,(\hat{\rho}|\rho = 0) = .007575$$

$$\frac{1}{2 + .6522n} = .007551$$

$$\text{Eff}\,(\hat{\rho}) = .6655$$

G. An Unbiased Estimator of ρ Using Four Pairs of Quantiles

Let z_i and z_i' $(i = 1, 2, 3, 4)$ be defined as in Part VII, paragraph D. Then an unbiased estimator of ρ using a linear combination of four pairs of symmetric transformer sample quantiles is given by

$$\hat{\rho} = \frac{\alpha\,[z_4^2 - (z_4')^2 + (z_1^2 - (z_1')^2)] + \beta\,[z_3^2 - (z_3')^2 + z_2^2 - (z_2')^2]}{4\,[\alpha\,(a_4^2 + \zeta_4^2) + \beta\,(a_3^2 + \zeta_3^2)]}$$

where

$$a_i^2 = \frac{F\,(\zeta_i)\,[1 - F\,(\zeta_i)]}{nf^2\,(\zeta_i)} \qquad (i = 3, 4)$$

Then

$$E(\hat{\rho}) = \frac{1}{4\left[\alpha\left(a_4^2 + \zeta_4^2\right) + \beta\left(a_3^2 + \zeta_3^2\right)\right]}\Big\{ 2\alpha\left[\zeta_4^2(1 + \rho) + a_4^2(1 + \rho) - \zeta_4^2(1 - \rho) - a_4^2(1 - \rho)\right]$$

$$+ 2\beta\left[\zeta_3^2(1 + \rho) + a_3^2(1 + \rho) - \zeta_3^2(1 - \rho) - a_3^2(1 - \rho)\right] = \rho$$

$$\text{Var}(\hat{\rho}) = \frac{1}{16\left[\alpha\left(a_4^2 + \zeta_4^2\right) + \beta\left(a_3^2 + \zeta_3^2\right)\right]^2}\Big\{ \alpha^2\left[\text{Var}\left(z_4^2 + z_1^2\right) + \text{Var}\left((z_4')^2 + (z_1')^2\right)\right]$$

$$+ \beta^2\left[\text{Var}\left(z_3^2 + z_2^2\right) + \text{Var}\left((z_3')^2 + (z_2')^2\right)\right]$$

$$+ 2\alpha\beta\left[\text{Cov}\left(z_4^2 + z_1^2, z_3^2 + z_2^2\right) + \text{Cov}\left((z_4')^2 + (z_1')^2, (z_3')^2 + (z_2')^2\right)\right]\Big\}$$

$$= \frac{1}{2\left[\alpha\left(a_4^2 + \zeta_4^2\right) + \beta\left(a_3^2 + \zeta_3^2\right)\right]^2}\Big\{ \alpha^2\left[a_4^4\left(1 + \rho_{14}^2\right) + 2a_4^2\zeta_4^2\left(1 - \rho_{14}\right)\right]$$

$$+ \beta^2\left[a_3^4\left(1 + \rho_{13}^2\right) + 2a_3^2\zeta_3^2\left(1 - \rho_{23}\right)\right]$$

$$+ 2\alpha\beta\left[2a_3a_4\zeta_3\zeta_4\left(\rho_{12} - \rho_{13}\right) + a_3^2a_4^2\left(\rho_{14}^2 - \rho_{13}^2\right)\right]\Big\}(1 + \rho^2)$$

where ρ_{ij} denotes the correlation between z_i and z_j. As in the previous cases, we will use the values of the parameters as given in Part VII, paragraph D. These are

$$\alpha = .116 \qquad \beta = .236$$
$$p_1 = .0230 \qquad p_2 = .1269$$
$$p_4 = .9770 \qquad p_3 = .8731$$

For these values:

$$\hat{\rho} = \frac{.116\left[z_4^2 - (z_4')^2 + z_1^2 - (z_1')^2\right] + .236\left[z_3^2 - (z_3')^2 + z_2^2 - (z_2')^2\right]}{\dfrac{5.930}{n} + 3.0754}$$

and for n = 200,

$$\hat{\rho} = .0374\left[z_4^2 - (z_4')^2 + z_1^2 - (z_1')^2\right]$$
$$+ .0760\left[z_3^2 - (z_3')^2 - z_2^2 - (z_2')^2\right]$$

$$\text{Var}(\hat{\rho}\,|\,\rho = 0) = .006188$$
$$\text{Eff}(\hat{\rho}) = .8121$$

This efficiency is quite high and makes this method of data compression very attractive.

It is of interest to compare the present method of estimating ρ from quantiles to a method proposed by F. Mosteller (Ref. 9) using bivariate order statistics. The technique he uses is to construct lines $y = 0$ and $x = \pm k$,

which cut the xy plane into six parts. The estimate of ρ is based on the number of pairs of observations falling in the four corners. Briefly, let

$n_1 =$ the number of pairs of observations (x_i, y_i)
    such that $(x_i > k, y_i > 0)$

$n_2 =$ the number of pairs of observations (x_i, y_i)
    such that $(x_i < -k, y_i > 0)$

$n_3 =$ the number of pairs of observations (x_i, y_i)
    such that $(x_i < -k, y_i < 0)$

$n_4 =$ the number of pairs of observations (x_i, y_i)
    such that $(x_i > k, y_i < 0)$

$$s_1 = \int_0^\infty \int_k^\infty g_3(x,y)\,dx\,dy$$

$$\lambda = \int_k^\infty f(x)\,dx$$

The maximum likelihood estimator $\hat{\rho}$ of ρ based on the n_i is then found by solving for $\hat{\rho}$ the equation

$$\frac{n_1 + n_3}{\sum\limits_{i=1}^4 n_i} = \left(\frac{s_1}{\lambda}\right)_{\rho = \hat{\rho}}$$

The $\mathrm{Var}\,(\hat{\rho}\,|\,\rho = 0)$ is a minimum for $k = .6121$ and is equal to $1.939/n$. Thus,

$$\mathrm{Eff}\,(\hat{\rho}) = \frac{n}{(n-1)(1.939)} = .5183 \quad \text{for} \quad n = 200$$

which is greater than the efficiency of $\hat{\rho}$ using one pair of quantiles but less than the efficiency of the estimators using more than one pair, as we have seen.

H. Application

An application of this method of data compression for computing correlations will now be given. Consider the problem of determining the correlation between micrometeoroid intensities measured at two different places on a deep-space probe. It can be assumed that from previous flights the mean and variance of the number of counts/sec are both known before the flight, or at any rate before the experiment. What is desired is a measure of the correlations between the counts/sec at two different positions and orientations of the counters on the spacecraft. This correlation gives a measure of the direction from which the micrometeoroids are arriving.

The spacecraft would need other equipment besides a Quantiler to perform this data compression. Extra equipment would be needed to form the linear combinations of the x_i and y_i to get the u_i and v_i. Such operations, however, are easy to perform. Data compression ratios on the order of 100 to 1 are usually obtainable with the use of four quantiles, with little or no loss of statistical efficiency. And, as usual, the quantile method requires less equipment than would be necessary to compute the sample correlation on board the spacecraft. The method of recovering the direction from the correlation is not considered here. In any case, such calculations are performed on the ground and do not affect the on-board equipment.[2]

VIII. APPLYING THE TESTS TO REAL DATA

Two sets of samples, each containing 200 sample values, were drawn from a table of random numbers (Ref. 10), in which the entries are independently distributed $N(0, 1)$. Hence, the sets of sample values can be considered as samples of two independent normal random variables x and y, respectively, with means $\mu_x = \mu_y = 0$ and variances $\sigma_x^2 = \sigma_y^2 = 1$. The sample quantiles (denoted by $z(p)$ and $z'(p)$, respectively) necessary to perform Tests A, B, D, E, and F, as well as those for the estimation of ρ, were determined. All the tests were performed at a significance level of .01. From the samples of x, the following values were obtained.

$$z(.5) = -.064 \quad z(.1068) = -1.230$$

$$z(.2703) = -.640 \quad z(.8932) = 1.218$$

$$z(.7297) = .596 \quad z(.3512) = -.345$$

$$z(.6488) = .371$$

$$z(.9424) = 1.621 \quad z(.023) = -2.159$$

$$z(.0690) = -1.400 \quad z(.977) = 2.360$$

$$z(.9310) = 1.477 \quad z(.127) = -1.183$$

$$z(.873) = 1.192$$

From the samples of y, the following values were obtained:

$$z'(.5) = .028 \quad z'(.1068) = -1.153$$

$$z'(.2703) = -.600 \quad z'(.8932) = 1.242$$

$$z'(.7297) = .626 \quad z'(.3512) = -.346$$

$$z'(.6488) = .415$$

$$z'(.9424) = 1.602 \quad z'(.023) = -1.871$$

$$z'(.0690) = -1.359 \quad z'(.977) = 2.041$$

$$z(.9310) = 1.494 \quad z'(.127) = -.956$$

$$z'(.873) = 1.095$$

The sample means, $\bar{x}$ and $\bar{y}$, and the sample standard deviations, s_x and s_y, as well as the corresponding esti-

mates using four optimum quantiles, were computed and found to be

$$\bar{x} = .0094 \quad s_x = 1.074$$

$$\bar{y} = .0556 \quad s_y = .982$$

$$\hat{\mu}_x = .0057 \quad \hat{\sigma}_x = 1.085$$

$$\hat{\mu}_y = .038 \quad \hat{\sigma}_y = .938$$

The estimators of $\rho = 0$ using one, two, and four quantiles (denoted by $\hat{\rho}_1$, $\hat{\rho}_2$, and $\hat{\rho}_4$) and the sample correlation r, were also computed and were found to be

$$\hat{\rho}_1 = .0122 \quad \hat{\rho}_4 = .1458$$

$$\hat{\rho}_2 = .0070 \quad r = .0525$$

Thus, one sees that in this case the poorest estimate was obtained using four quantiles and the next poorest estimate was the one obtained using all the sample values. This result is not inconsistent since, asymptotically, $\sigma_r = .071$ and $\sigma_{\hat{\rho}_1} = .079$.

Tests A and B using one, two, and four sample quantiles were performed independently on both sets of samples, with H_0 being true. In all twelve tests, H_0 was accepted. For Tests D and E, which required sample quantiles from both sets of samples for each test, H_0 was accepted in all six tests when H_0 was true. For Test F, it was assumed that the given sets of sample values were actually transformed values $\{u_i\}$ and $\{v_i\}$ obtained from sets $\{x_i\}$ and $\{y_i\}$ taken from two standard normal distributions, with $\rho = 0$. With H_0 being true, in each of the three tests of Test F, H_0 was accepted.

Now, if x is distributed $N(\mu, \sigma)$, then $x' = ax + b$, $a > 0$, is distributed $N(a\mu + b, a\sigma)$. If the above transformation were applied to all the sample values taken from a population distributed $N(\mu, \sigma)$, one sees that not only would the new sample values be distributed $N(a\mu + b, a\sigma)$, but the order of the samples would remain unchanged; that is, if $x_i < x_j$, then $x'_i < x'_j$. Hence, if $z(p)$ were the quantile of order p of the $\{x_i\}$, then $az(p) + b$ would be the quantile of order p of the $\{x'_i\}$. This fact permits us to perform Tests A, B, D, E, and F when H_0 is *not true* by simply performing a linear transformation on the sample quantiles of the x_i and y_i. These tests will be given in detail. The best tests using all the sample values will also be given.

In Test A, by adding .2 to each quantile $z(p)$ and $z'(p)$, one can assume in each case that $\mu_1 = 0$, $\mu_2 = .2$, $\sigma = 1$, and H_1 is true. The result of each test and the decision are as follows ($\tilde{z}(p)$ and $\tilde{z}'(p)$ will denote the values of the sample quantiles *after* the transformation):

$$\tilde{z}(.5) = .136 < .2061, \quad \text{accept } H_0$$

$$\tilde{z}'(.5) = .228 > .2061, \quad \text{reject } H_0$$

$$\tilde{z}(.2703) + \tilde{z}(.7297) = .356 < .3655, \quad \text{accept } H_0$$

$$\tilde{z}'(.2703) + \tilde{z}(.7297) = .426 > .3655, \quad \text{reject } H_0$$

$$.192\,[\tilde{z}(.1068) + \tilde{z}(.8932)] + .308\,[\tilde{z}(.3512)$$
$$+ \tilde{z}(.6488)] = .2057 > .1720, \quad \text{reject } H_0$$

$$.192\,[\tilde{z}'(.1068) + \tilde{z}(.8932)] + .308\,[\tilde{z}'(.3512)$$
$$+ \tilde{z}'(.6488)] = .2383 > .1720, \quad \text{reject } H_0$$

Adding .2 to each sample value and then applying Test A to all the sample values results in

$$\frac{1}{n}\sum_{i=1}^{n} x_i = .2094 > .1645, \quad \text{reject } H_0$$

$$\frac{1}{n}\sum_{i=1}^{n} y_i = .2556 > .1645, \quad \text{reject } H_0$$

In Test B, by multiplying each quantile $z(p)$ and $z'(p)$ by 1.2, one can assume in each case that $\mu = 0$, $\sigma_1 = 1.0$, $\sigma_2 = 1.2$, and H_1 is true. Then one has

$$\tilde{z}(.9424) = 1.945 > 1.907, \quad \text{reject } H_0$$

$$\tilde{z}'(.9424) = 1.922 > 1.907, \quad \text{reject } H_0$$

$$\tilde{z}(.9310) - \tilde{z}(.0690) = 3.452 > 3.393, \quad \text{reject } H_0$$

$$\tilde{z}'(.9310) - \tilde{z}'(.0690) = 3.424 > 3.393, \quad \text{reject } H_0$$

$$.116\,[\tilde{z}(.977) - \tilde{z}(.023)] + .236\,[\tilde{z}(.873)$$
$$- \tilde{z}(.127)] = 1.303 > 1.129, \quad \text{reject } H_0$$

$$.116\,[\tilde{z}'(.977) - \tilde{z}(.023)] + .236\,[\tilde{z}'(.873)$$
$$- \tilde{z}'(.127)] = 1.125 < 1.129, \quad \text{accept } H_0$$

Multiplying each sample value by 1.2 and then applying Test B to all the sample values results in

$$\left(2\sum_{i=1}^{n} x_i^2\right)^{1/2} = 25.78 > 22.326, \quad \text{reject } H_0$$

$$\left(2\sum_{i=1}^{n} y_i^2\right)^{1/2} = 23.62 > 22.326, \quad \text{reject } H_0$$

In Test D, by putting $\theta = .25$ and hence adding .25 to each $z'(p)$ and leaving each $z(p)$ unchanged, one can assume that $\theta = .25$, $\mu_2 = \mu_1 + .25$, $\sigma_1 = \sigma_2 = 1$ and H_1 is true. Then one has

$$z(.5) - \tilde{z}'(.5) = -.342 < -.2915, \quad \text{reject } H_0$$

$$z(.7297) - \tilde{z}'(.7297) + z(.2703)$$
$$- \tilde{z}'(.2703) = -.569 < -.5169, \quad \text{reject } H_0$$

$$.198\,[z(.1068) - \tilde{z}'(.1068) + z(.8932) - \tilde{z}'(.8932)]$$
$$+ .308\,[z(.3512) - \tilde{z}'(.3512) + z(.6488) - \tilde{z}'(.6488)]$$
$$= -.2826 < -.2433, \quad \text{reject } H_0$$

Adding .25 to each y_i and leaving each x_i unchanged, and then applying Test D to all the sample values results in

$$\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n} y_i = -.2962 < -.2326, \quad \text{reject } H_0$$

In Test E, by multiplying each $z'(p)$ by 1.25 and leaving each $z(p)$ unchanged, one can assume that $\theta = 1.25$, $\mu_1 = \mu_2 = 0$, $\sigma_2 = 1.25\sigma$, and H_1 is true. Then one has

$$\frac{z(.9424)}{\tilde{z}'(.9424)} = .809 > .738, \quad \text{accept } H_0$$

$$\frac{z(.9310) - z(.0690)}{\tilde{z}'(.9310) - \tilde{z}'(.0690)} = .807 < .814, \quad \text{reject } H_0$$

$$\frac{.116\,[z(.9770) - z(.0230)] + .236\,[z(.8731) - z(.1269)]}{.116\,[\tilde{z}'(.9770) - \tilde{z}'(.0230)] + .236\,[\tilde{z}'(.8731) - \tilde{z}'(.1269)]}$$
$$= .925 > .834, \quad \text{accept } H_0$$

Multiplying each y_i by 1.25 and leaving each x_i unchanged, and then applying Test E to all the sample values results in

$$\frac{1}{2}\ln\left(\frac{\displaystyle\sum_{i=1}^{n} x_i^2}{\displaystyle\sum_{i=1}^{n} y_i^2}\right) = -.1354 > -.1645, \quad \text{accept } H_0$$

In Test F, by multiplying each $z(p)$ by $(1.2)^{1/2}$ and each $z'(p)$ by $(.8)^{1/2}$, it can be assumed that each resulting $z(p)$ is the quantile of order p of a transformed set of samples $\{u_i\}$ distributed $N[0, (1 + \rho)^{1/2}]$, and each resulting $z'(p)$ is the quantile of order p of a transformed set $\{v_i\}$, distributed $N[0, (1 - \rho)^{1/2}]$, and the transformations were applied to correlated sets $\{x_i\}$ and $\{y_i\}$, each distributed $N(0,1)$ with $\rho = .2$. Hence, for Test F one has $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$, $\rho = .2$, and H_1 is true. The results of the tests are as follows:

$$\tilde{z}(.9424) - \tilde{z}'(.9424) = .343 < .4697, \quad \text{accept } H_0$$

$$\tilde{z}(.9310) - \tilde{z}'(.9310) - \tilde{z}(.0690) + \tilde{z}'(.0690)$$
$$= .599 < .6041, \quad \text{accept } H_0$$

$$.116\,[\tilde{z}(.9770) - \tilde{z}'(.9770) - \tilde{z}(.0230) + \tilde{z}'(.0230)]$$
$$+ .236\,[\tilde{z}(.8731) - \tilde{z}'(.8731) - \tilde{z}(.1269) + \tilde{z}'(.1269)]$$
$$= .3494 > .1814, \quad \text{reject } H_0$$

Multiplying each x_i by $(1.2)^{1/2}$ and each y_i by $(.8)^{1/2}$, and then applying Test F to all the sample values results in

$$\sum_{i=1}^{n}(\rho - 1)\,u_i^2 + \sum_{i=1}^{n}(\rho + 1)\,v_i^2 = -3.563 < 12.91, \quad \text{reject } H_0$$

Tables 2a and 2b give the results of applying the single quantile test of Test C to the two sets of samples for various values of σ_2 and μ_2/σ_2 ($\mu_1 = 0$, $\sigma_1 = 1$), as shown in Columns 1 and 2 of the tables. The quantiles used in each test were determined by using the optimum values of p shown in Column 4, and are given as $z(p)$ and $z'(p)$. For these values of $z(p)$ and $z'(p)$, H_0 is true and the decisions given in Columns 7 and 9 are obtained by comparing $z(p)$ and $z'(p)$ with the values of δ in Column 5. The decisions were made according to the acceptance criteria given in Tables 1a and 1b.

Table 2a. Applying Test C using one optimum quantile, for $\sigma_2 > \sigma_1$, $\varepsilon = .01$, $n = 200$

1 σ_2	2 μ_2/σ_2	3 μ_2	4 Opt p	5 δ	6 $z(p)$	7 H_0 true decision	8 $z'(p)$	9 H_0 true decision	10 $\tilde{z}(p)$	11 H_1 true decision	12 $\tilde{z}'(p)$	13 H_1 true decision
1.025	.05	.051	.8133	1.129	.975	accept H_0	.776	accept H_0	1.050	accept H_0	.846	accept H_0
1.025	.10	.103	.7190	.799	.540		.569		.657	↓	.686	↓
1.025	.15	.154	.6628	.633	.436		.445		.601		.610	
1.025	.20	.205	.6255	.530	.331		.382		.544	reject H_0	.597	reject H_0
1.025	.25	.256	.6026	.469	.272		.305		.535		.569	
1.025	.30	.308	.5871	.428	.256		.290		.570	↓	.605	↓
1.025	.35	.359	.5754	.398	.232		.247		.597	↓	.612	↓
1.05	.05	.053	.8790	1.437	1.194		1.145		1.307	accept H_0	1.255	accept H_0
1.05	.10	.105	.8133	1.129	.975		.776		1.129	reject H_0	.920	↓
1.05	.15	.158	.7580	.926	.781		.678		.978	↓	.870	↓
1.05	.20	.210	.7158	.789	.540		.569		.777	accept H_0	.807	reject H_0
1.05	.25	.263	.6844	.695	.492		.488		.780	reject H_0	.775	
1.05	.30	.315	.6591	.623	.386		.437		.720	↓	.774	↓
1.05	.35	.368	.6368	.561	.345		.401		.730	↓	.789	↓
1.10	.05	.055	.9115	1.641	1.372		1.372		1.564	accept H_0	1.564	accept H_0
1.10	.10	.110	.8749	1.414	1.192		1.095		1.421	reject H_0	1.315	↓
1.10	.15	.165	.8389	1.238	1.066		.893		1.338	↓	1.147	↓
1.10	.20	.220	.8051	1.096	.937		.769		1.251	↓	1.066	↓
1.10	.30	.330	.7518	.904	.767		.672		1.174	↓	1.069	reject H_0
1.10	.35	.385	.7291	.831	.596		.626		1.041	↓	1.074	↓
1.15	.05	.058	.9207	1.711	1.388		1.437		1.654	accept H_0	1.711	
1.15	.10	.115	.8962	1.538	1.220		1.317		1.518	↓	1.630	
1.15	.20	.230	.8461	1.270	1.074		.951		1.465	reject H_0	1.324	
1.15	.30	.345	.7996	1.076	.882		.763		1.359	↓	1.222	
1.20	.05	.060	.9251	1.746	1.405		1.473		1.746		1.828	
1.20	.10	.120	.9066	1.607	1.365		1.326		1.758		1.711	
1.20	.20	.240	.8686	1.381	1.166		1.081		1.639		1.537	
1.20	.25	.300	.8485	1.281	1.074	↓	.951	↓	1.589	↓	1.441	↓

Table 2b. Applying Test C using one optimum quantile, for $\sigma_2 < \sigma_1$, $\varepsilon = .01$, $n = 200$

1 σ_2	2 μ_2/σ_2	3 μ_2	4 Opt p	5 δ	6 z(p)	7 H_0 true decision	8 z'(p)	9 H_0 true decision	10 $\tilde{z}(p)$	11 H_1 true decision	12 $\tilde{z}'(p)$	13 H_1 true decision
.95	.05	.048	.1170	.921	−1.195	accept H_0	−1.083	accept H_0	−1.087	accept H_0	−.981	accept H_0
.95	−.05	−.048	.8830	.921	1.197		1.168		1.089		.987	↓
.95	.10	.095	.1762	.688	−.993		−.810		−.848		−.675	reject H_0
.95	−.10	−.095	.8238	.688	1.006		.850		.861		.713	accept H_0
.95	.20	.190	.2709	.389	−.640		−.600		−.418	↓	−.380	reject H_0
.95	−.20	−.190	.7291	.389	.596		.626		.376	reject H_0	.405	accept H_0
.95	.30	.285	.3300	.226	−.446		−.398		−.139		−.093	reject H_0
.95	−.30	−.285	.6700	.226	.482		.478		.173		.169	↓
.95	.40	.380	.3669	.129	−.295		−.318		.100		.078	
.95	−.40	−.380	.6331	.129	.341		.391		−.056	↓	−.009	↓
.90	.05	.045	.0838	1.084	−1.302		−1.263		−1.127	accept H_0	−1.092	accept H_0
.90	−.05	−.045	.9162	1.084	1.374		1.381		1.192		1.198	↓
.90	.10	.090	.1112	.947	−1.216		−1.149		−1.004		−.944	reject H_0
.90	−.10	−.090	.8888	.947	1.205		1.179		.995		.971	accept H_0
.90	.20	.180	.1685	.715	−1.030		−.823		−.747	↓	−.561	reject H_0
.90	−.20	−.180	.8315	.715	1.044		.874		.760		.607	
.90	.30	.270	.2206	.540	−.739		−.722		−.395	reject H_0	−.380	
.90	−.30	−.270	.7794	.540	.856		.689		.500		.350	
.90	.35	.315	.2420	.474	−1.030		−.676		−.309		−.293	
.90	−.35	−.315	.7580	.474	.781		.678		.388		.295	
.80	.05	.040	.0681	1.175	−1.400		−1.359		−1.080		−1.047	
.80	−.05	−.040	.9319	1.175	1.477		1.494		1.142		1.155	
.80	.10	.080	.0808	1.101	−1.302		−1.263		−.962		−.930	
.80	−.10	−.080	.9192	1.101	1.374		1.381		1.019		1.025	
.80	.20	.160	.1056	.973	−1.230		−1.153		−.824		−.762	
.80	−.20	−.160	.8944	.973	1.218		1.242		.814		.834	
.80	.30	.240	.1314	.859	−1.161		−.939		−.689		−.511	
.80	−.30	−.240	.8686	.859	1.166		1.081		.693		.625	
.75	.05	.038	.0655	1.191	−1.400		−1.359		−1.012		−.981	
.75	−.05	−.038	.9345	1.191	1.477		1.494		1.090		1.083	
.75	.10	.075	.0749	1.134	−1.361		−1.331		−.946		−.923	
.75	−.10	−.075	.9251	1.134	1.405		1.473		.979		1.029	
.75	.15	.113	.0838	1.084	−1.302		−1.263		−.864		−.834	
.75	−.15	−.113	.9162	1.084	1.374	↓	1.381	↓	.918	↓	.923	↓

The values of $\tilde{z}(p)$ and $\tilde{z}'(p)$ were determined by the relationships

$$\tilde{z}(p) = \sigma_2 z(p) + \mu_2$$

$$\tilde{z}'(p) = \sigma_2 z'(p) + \mu_2$$

By using the transformed values $\tilde{z}(p)$ and $\tilde{z}'(p)$ in the tests, it can be assumed that H_1 is true. The decisions in these cases are given in Columns 11 and 13. It can be seen that when H_0 is true, the null hypothesis is accepted in all cases, which is not surprising for $\varepsilon = .01$. Moreover, as one might expect, when H_1 is true, H_0 is rejected most of the time when P_0 is high and accepted most of the time when P_0 is low.

IX. SUBOPTIMUM TEST STATISTICS

Tables 3 through 7 give the test statistics and acceptance regions to be used in Tests A, B, D, E, and F using one, two, and four quantiles. With the exception of Test E, which is given for $\varepsilon = .01$ and $n = 200$, the tests are given as functions of n for both $\varepsilon = .01$ and $\varepsilon = .05$.

However, in order to apply the results developed here to statistical experiments performed aboard a spacecraft, it is necessary to specify the order of the quantiles in advance. For maximum data compression, only one set of k quantiles should be so specified for a k quantile test or estimator, regardless of which test or estimator is required. The problem then, of course, is to decide on which set of k quantiles to use. Since a set of quantiles which is optimum for one test is not, as we have seen, necessarily

optimum for another, it is obvious that a compromise is required based on some reasonable criterion. This problem is not a new one; it was encountered in our previous investigations into the use of quantiles for data compression. Hence a proposed solution is at hand and will be presented here, but will be restricted to the four-quantile case.

Table 3. Test statistics and acceptance regions for Test A

$H_0: \quad g(x) = g_1(x) = N(\mu_1, \sigma)$
$H_1: \quad g(x) = g_2(x) = N(\mu_2, \sigma)$ $\quad \sigma$ known

Conditions	Acceptance regions
$\mu_2 > \mu_1$ $\varepsilon = .01$	$z(.5) < \mu_1 + \dfrac{2.915\sigma}{n^{1/2}}$ $z(.2703) + z(.7297) < 2\mu_1 + \dfrac{5.169\sigma}{n^{1/2}}$ $.192\,[z(.1068) + z(.8932)] + .308\,[z(.3512) + z(.6488)]$ $< \mu_1 + \dfrac{2.433\sigma}{n^{1/2}}$
$\mu_2 > \mu_1$ $\varepsilon = .05$	$z(.5) < \mu_1 + \dfrac{2.062\sigma}{n^{1/2}}$ $z(.2703) + z(.7297) < 2\mu_1 + \dfrac{3.656\sigma}{n^{1/2}}$ $.192\,[z(.1068) + z(.8932)] + .308\,[z(.3512) + z(.6488)]$ $< \mu_1 + \dfrac{1.721\sigma}{n^{1/2}}$
$\mu_2 < \mu_1$ $\varepsilon = .01$	$z(.5) > \mu_1 - \dfrac{2.915\sigma}{n^{1/2}}$ $z(.2703) + z(.7297) > 2\mu_1 - \dfrac{5.169\sigma}{n^{1/2}}$ $.192\,[z(.1068) + z(.8932)] + .308\,[z(.3512) + z(.6488)]$ $> \mu_1 - \dfrac{2.433\sigma}{n^{1/2}}$
$\mu_2 < \mu_1$ $\varepsilon = .05$	$z(.5) > \mu_1 - \dfrac{2.062\sigma}{n^{1/2}}$ $z(.2703) + z(.7297) > 2\mu_1 - \dfrac{3.656\sigma}{n^{1/2}}$ $.192\,[z(.1068) + z(.8932)] + .308\,[z(.3512) + z(.6488)]$ $> \mu_1 - \dfrac{1.721\sigma}{n^{1/2}}$

Table 4. Test statistics and acceptance regions for Test B

$H_0: \quad g(x) = g_1(x) = N(\mu, \sigma_1)$
$H_1: \quad g(x) = g_2(x) = N(\mu, \sigma_2)$ $\quad \mu$ known

Conditions	Acceptance regions
$\sigma_2 > \sigma_1$ $\varepsilon = .01$	$z(.9424) < \mu + \sigma_1\left(1.575 + \dfrac{4.697}{n^{1/2}}\right)$ $z(.0576) > \mu - \sigma_1\left(1.575 + \dfrac{4.697}{n^{1/2}}\right)$ $z(.9310) - z(.0690) < \sigma_1\left(2.966 + \dfrac{6.041}{n^{1/2}}\right)$ $.116\,[z(.977) - z(.023)] + .236\,[z(.873) - z(.127)]$ $< \sigma_1\left(1.001 + \dfrac{1.814}{n^{1/2}}\right)$
$\sigma_2 > \sigma_1$ $\varepsilon = .05$	$z(.9424) < \mu + \sigma_1\left(1.575 + \dfrac{3.322}{n^{1/2}}\right)$ $z(.0576) > \mu - \sigma_1\left(1.575 + \dfrac{3.322}{n^{1/2}}\right)$ $z(.9310) - z(.0690) < \sigma_1\left(2.966 + \dfrac{4.272}{n^{1/2}}\right)$ $.116\,[z(.977) - z(.023)] + .236\,[z(.873) - z(.127)]$ $< \sigma_1\left(1.001 + \dfrac{1.283}{n^{1/2}}\right)$
$\sigma_2 < \sigma_1$ $\varepsilon = .01$	$z(.9424) > \mu + \sigma_1\left(1.575 - \dfrac{4.697}{n^{1/2}}\right)$ $z(.0576) < \mu - \sigma_1\left(1.575 - \dfrac{4.697}{n^{1/2}}\right)$ $z(.9310) - z(.0690) > \sigma_1\left(2.966 - \dfrac{6.041}{n^{1/2}}\right)$ $.116\,[z(.977) - z(.023)] + .236\,[z(.873) - z(.127)]$ $> \sigma_1\left(1.001 - \dfrac{1.814}{n^{1/2}}\right)$
$\sigma_2 < \sigma_1$ $\varepsilon = .05$	$z(.9424) > \mu + \sigma_1\left(1.575 - \dfrac{3.322}{n^{1/2}}\right)$ $z(.0576) < \mu - \sigma_1\left(1.575 - \dfrac{3.322}{n^{1/2}}\right)$ $z(.9310) - z(.0690) > \sigma_1\left(2.966 - \dfrac{4.272}{n^{1/2}}\right)$ $.116\,[z(.977) - z(.023)] + .236\,[z(.873) - z(.127)]$ $> \sigma_1\left(1.001 - \dfrac{1.283}{n^{1/2}}\right)$

Table 5. Test statistics and acceptance regions for Test D

H_0: $g_1(x) = N(\mu, \sigma)$ $\quad$ $g_2(y) = N(\mu, \sigma)$, $\quad$ σ known, μ unknown

H_1: $g_1(x) = N(\mu, \sigma)$ $\quad$ $g_2(y) = N(\mu + \theta, \sigma)$, $\quad$ $\theta \neq 0$

Conditions	Acceptance regions
$\theta > 0$ $\varepsilon = .01$	$z(.5) - z'(.5) > -2.915\sigma \left(\frac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$ $z(.7297) - z'(.7297) + z(.2703) - z'(.2703)$ $> -5.1695 \left(\frac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$ $.192 [z(.1068) - z'(.1068) + z(.8932) - z'(.8932)]$ $+ .308 [z(.3512) - z'(.3512) + z(.6488) - z'(.6488)]$ $> -2.433\sigma \left(\frac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$
$\theta > 0$ $\varepsilon = .05$	$z(.5) - z'(.5) > -2.062\sigma \left(\frac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$ $z(.7297) - z'(.7297) + z(.2703) - z'(2703)$ $> -3.656\sigma \left(\frac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$ $.192 [z(.1068) - z'(.1068) + z(.8932) - z'(.8932)]$ $+ .308 [z(.3512) - z'(.3512) + z(.6488) - z'(.6488)]$ $> -1.721\sigma \left(\frac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$
$\theta < 0$ $\varepsilon = .01$	$z(.5) - z'(.5) < 2.915\sigma \left(\frac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$ $z(.7297) - z'(.7297) + z(.2703) - z'(2703)$ $< 5.169\sigma \left(\frac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$ $.192 [z(.1068) - z'(.1068) + z(.8932) - z'(.8932)]$ $+ .308 [z(.3512) - z'(.3512) + z(.6488) - z'(.6488)]$ $< 2.433\sigma \left(\frac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$
$\theta < 0$ $\varepsilon = .05$	$z(.5) - z'(.5) < 2.062\sigma \left(\frac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$ $z(.7297) - z'(.7297) + z(.2703) - z'(2703)$ $< 3.656\sigma \left(\frac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$ $.192 [z(.1068) - z'(.1068) + z(.8932) - z'(.8932)]$ $+ .308 [z(.3512) - z'(.3512) + z(.6488) - z'(.6488)]$ $< 1.721\sigma \left(\frac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$

Table 6. Test statistics and acceptance regions for Test E

H_0: $g_1(x) = N(0, \sigma)$, $\quad$ $g_2(y) = N(0, \sigma)$

H_1: $g_1(x) = N(0, \sigma)$, $\quad$ $g_2(y) = N(0, \theta\sigma)$, $\quad$ $\theta > 0$

Conditions	Acceptance regions
$\theta > 1$ $\varepsilon = .01$ $n = 200$	$\frac{z(.9424)}{z'(.9424)} > .738$ $\frac{z(.9310) - z(.0690)}{z'(.9310) - z'(.0690)} > .814$ $\frac{.116 [z(.977) - z(.023)] + .236 [z(.873) - z(.127)]}{.116 [z'(.977) - z'(.023)] + .236 [z'(.873) - z'(.127)]}$ $> .834$
$\theta < 1$ $\varepsilon = .01$ $n = 200$	$\frac{z(.9424)}{z'(.9424)} < 1.354$ $\frac{z(.9310) - z(.0690)}{z'(.9310) - z'(.0690)} < 1.228$ $\frac{.116 [z(.977) - z(.023)] + .236 [z(.873) - z(.127)]}{.116 [z'(.977) - z'(.023)] + .236 [z'(.873) - z'(.127)]}$ < 1.200

In the four-quantile cases, the weights α and β are also identical with those used in the estimators of μ and σ. It should not be surprising that, except for Test E, it was found that the set of quantiles which minimizes the variance of the *estimate* of μ or σ is identical with that which maximizes the power of the one-sided test concerned with the *value* of μ or σ. And with respect to Test E, which is a test on σ, it is felt that it is not unreasonable to assume that the same result holds. Thus we are faced with the problem of effecting a compromise between two sets of quantiles: one which minimizes $\text{Var}(\hat{\mu})$, and another which minimizes $\text{Var}(\hat{\sigma})$. The compromise we now propose for $k = 4$ is one which was adopted previously for estimating μ and σ. Determine the orders of the set of two pairs of symmetric quantiles and weights α_1, β_1 and α_2, β_2, such that unbiased estimators of μ and σ are given by

$$\hat{\mu} = \alpha_1 (z_1 + z_4) + \beta_1 (z_2 + z_3)$$

$$\hat{\sigma} = \alpha_2 (z_4 - z_1) + \beta_2 (z_3 - z_2)$$

and for which the linear combination $\text{Var}(\hat{\mu}) + b\,\text{Var}(\hat{\mu})$ is a minimum, $b = 1, 2, \cdots, .$

The same sets of quantiles are to be used in all tests, except in Test C, and for estimating ρ. Weights α_1 and β_1 are to be used in the test statistics of Tests A and D, while weights α_2 and β_2 are to be used in the test statistics of Tests B, E, and F and in estimating ρ.

It has no doubt been noted that, with the exception of Test C, only two sets of k quantiles have been used for the tests and for estimating ρ, for $k = 1, 2, 4$. The sets used in Test A and D are those which provide the asymptotically unbiased estimators of the mean of a single normal population with minimum variance, and the sets used in Tests B, E, and F and for estimating ρ are those which provide the asymptotically unbiased estimators of the standard deviation with minimum variance.

In Ref. 1, sets of four quantiles are given which meet the above conditions for $b = 1, 2, 3$. The orders of the quantiles and the weights are as follows:

For $b = 1$:

$$\alpha_1 = .141 \qquad \beta_1 = .359$$

$$\alpha_2 = .258 \qquad \beta_2 = .205$$

$$p_1 = .0668 \qquad p_2 = .2912$$

$$p_4 = .9332 \qquad p_3 = .7088$$

For $b = 2$:

$$\alpha_1 = .106 \qquad \beta_1 = .394$$

$$\alpha_2 = .196 \qquad \beta_2 = .232$$

$$p_1 = .0434 \qquad p_2 = .2381$$

$$p_4 = .9566 \qquad p_3 = .7619$$

Table 8. Suboptimum test statistics and acceptance regions for $k = 4$, $b = 1$

$$p_1 = .0668 \qquad p_2 = .2912$$
$$p_4 = .9332 \qquad p_3 = .7088$$

Conditions	Acceptance regions
$\mu_2 > \mu_1$, $\varepsilon = .01$ $\mu_2 > \mu_1$, $\varepsilon = .05$ $\mu_2 < \mu_1$, $\varepsilon = .01$ $\mu_2 < \mu_1$, $\varepsilon = .05$	**Test A:** $y = .141 (z_1 + z_4) + .359 (z_2 + z_3)$ $y < \mu_1 + 2.441\sigma/n^{1/2}$ $y < \mu_1 + 1.726\sigma/n^{1/2}$ $y > \mu_1 - 2.441\sigma/n^{1/2}$ $y > \mu_1 - 1.726\sigma/n^{1/2}$
$\sigma_2 > \sigma_1$, $\varepsilon = .01$ $\sigma_2 > \sigma_1$, $\varepsilon = .05$ $\sigma_2 < \sigma_1$, $\varepsilon = .01$ $\sigma_2 < \sigma_1$, $\varepsilon = .05$	**Test B:** $y = .258 (z_4 - z_1) + .205 (z_3 - z_2)$ $y < \sigma_1 (1.000 + 1.917/n^{1/2})$ $y < \sigma_1 (1.000 + 1.356/n^{1/2})$ $y > \sigma_1 (1.000 - 1.917/n^{1/2})$ $y > \sigma_1 (1.000 - 1.356/n^{1/2})$
$\theta > 0$, $\varepsilon = .01$ $\theta > 0$, $\varepsilon = .05$ $\theta < 0$, $\varepsilon = .01$ $\theta < 0$, $\varepsilon = .05$	**Test D:** $y = .141 (z_1 - z_1' + z_4 - z_4')$ $\qquad + .359 (z_2 - z_2' + z_3 - z_3')$ $y > -2.441\sigma \left(\dfrac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$ $y > -1.726\sigma \left(\dfrac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$ $y < 2.441\sigma \left(\dfrac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$ $y < 1.726\sigma \left(\dfrac{n_1 + n_2}{n_1 n_2} \right)^{1/2}$
$\rho > 0$, $\varepsilon = .01$ $\rho > 0$, $\varepsilon = .05$ $\rho < 0$, $\varepsilon = .01$ $\rho < 0$, $\varepsilon = .05$	**Test F:** $y = .258 (z_4 - z_4' - z_1' + z_1')$ $\qquad + .205 (z_3 - z_3' - z_2 + z_1')$ $y < 2.711/n^{1/2}$ $y < 1.917/n^{1/2}$ $y > -2.711/n^{1/2}$ $y > -1.917/n^{1/2}$

Table 7. Test statistics and acceptance regions for Test F

H_0: $g_1(x) = N(0, 1)$, $\quad g_2(y) = N(0, 1)$, $\quad \rho = 0$
H_1: $g_1(x) = N(0, 1)$, $\quad g_2(y) = N(0, 1)$, $\quad \rho \neq 0$

Conditions	Acceptance regions
$\rho > 0$ $\varepsilon = .01$	$z(.9424) - z'(.9424) < \dfrac{6.642}{n^{1/4}}$ $z(.9310) - z'(.9310) - z(.0690) + z'(.0690) < \dfrac{8.543}{n^{1/4}}$ $.116 [z(.9770) - z'(.9770) - z(.0230) + z'(.0230)]$ $+ .236 [z(.8731) - z'(.8731) - z(.1269) + z'(.1269)]$ $< \dfrac{2.565}{n^{1/4}}$
$\rho > 0$ $\varepsilon = .05$	$z(.9424) - z'(.9424) < \dfrac{4.697}{n^{1/4}}$ $z(.9310) - z'(.9310) - z(.0690) + z'(.0690) < \dfrac{6.042}{n^{1/4}}$ $.116 [z(.9770) - z'(.9770) - z(.0230) + z'(.0230)]$ $+ .236 [z(.8731) - z'(.8731) - z(.1269) + z'(.1269)]$ $< \dfrac{1.814}{n^{1/4}}$
$\rho < 0$ $\varepsilon = .01$	$z(.9424) - z'(.9424) > - \dfrac{6.642}{n^{1/4}}$ $z(.9310) - z'(.9310) - z(.0690) + z'(.0690)$ $> - \dfrac{8.543}{n^{1/4}}$ $.116 [z(.9770) - z'(.9770) - z(.0230) + z'(.0230)]$ $+ .236 [z(.8731) - z'(.8731) - z(.1269) + z'(.1269)]$ $> - \dfrac{2.565}{n^{1/4}}$
$\rho < 0$ $\varepsilon = .05$	$z(.9424) - z'(.9424) > - \dfrac{4.697}{n^{1/4}}$ $z(.9310) - z'(.9310) - z(.0690) + z'(.0690)$ $> - \dfrac{6.042}{n^{1/4}}$ $.116 [z(.9770) - z'(.9770) - z(.0230) + z'(.0230)]$ $+ .236 [z(.8731) - z'(.8731) - z(.1269) + z'(.1269)]$ $> - \dfrac{1.814}{n^{1/4}}$

For b = 3:

$$\alpha_1 = .097 \qquad \beta_1 = .403$$
$$\alpha_2 = .179 \qquad \beta_2 = .235$$
$$p_1 = .0389 \qquad p_2 = .2160$$
$$p_4 = .9611 \qquad p_3 = .7840$$

New test statistics and acceptance regions using the suboptimum sets of quantiles were computed for Tests A, B, D, and F and are given in Tables 8 through 10. The power

Table 9. Suboptimum test statistics and acceptance regions for k = 4, b = 2

$$p_1 = .0434 \qquad p_2 = .2381$$
$$p_4 = .9566 \qquad p_3 = .7619$$

Conditions	Acceptance regions
	Test A: $y = .106\,(z_1 + z_4) + .394\,(z_2 + z_3)$
$\mu_2 > \mu_1, \quad \varepsilon = .01$	$y < \mu_1 + 2.485\sigma/n^{\frac{1}{2}}$
$\mu_2 > \mu_1, \quad \varepsilon = .05$	$y < \mu_1 + 1.757\sigma/n^{\frac{1}{2}}$
$\mu_2 < \mu_1, \quad \varepsilon = .01$	$y > \mu_1 - 2.485\sigma/n^{\frac{1}{2}}$
$\mu_2 < \mu_1, \quad \varepsilon = .05$	$y > \mu_1 - 1.757\sigma/n^{\frac{1}{2}}$
	Test B: $y = .196\,(z_4 - z_1) + .232\,(z_3 - z_1)$
$\sigma_2 > \sigma_1, \quad \varepsilon = .01$	$y < \sigma_1(1.002 + 1.857/n^{\frac{1}{2}})$
$\sigma_2 > \sigma_1, \quad \varepsilon = .05$	$y < \sigma_1(1.002 + 1.313/n^{\frac{1}{2}})$
$\sigma_2 < \sigma_1, \quad \varepsilon = .01$	$y > \sigma_1(1.002 - 1.875/n^{\frac{1}{2}})$
$\sigma_2 < \sigma_1, \quad \varepsilon = .05$	$y > \sigma_1(1.002 - 1.313/n^{\frac{1}{2}})$
	Test D: $y = .106\,(z_1 - z_1' + z_4 - z_4')$ $\quad + .394\,(z_2 - z_2' + z_3 - z_3')$
$\theta > 0, \quad \varepsilon = .01$	$y > -2.484\sigma\left(\dfrac{n_1 + n_2}{n_1 n_2}\right)^{\frac{1}{2}}$
$\theta > 0, \quad \varepsilon = .05$	$y > -1.757\sigma\left(\dfrac{n_1 + n_2}{n_1 n_2}\right)^{\frac{1}{2}}$
$\theta < 0, \quad \varepsilon = .01$	$y < 2.484\sigma\left(\dfrac{n_1 + n_2}{n_1 n_2}\right)^{\frac{1}{2}}$
$\theta < 0, \quad \varepsilon = .05$	$y < 1.757\sigma\left(\dfrac{n_1 + n_2}{n_1 n_2}\right)^{\frac{1}{2}}$
	Test F: $y = .196\,(z_4 - z_4' - z_1 + z_1')$ $\quad + .232\,(z_3 - z_3' - z_2 + z_2')$
$\rho > 0, \quad \varepsilon = .01$	$y < 2.652/n^{\frac{1}{2}}$
$\rho > 0, \quad \varepsilon = .05$	$y < 1.876/n^{\frac{1}{2}}$
$\rho < 0, \quad \varepsilon = .01$	$y > -2.652/n^{\frac{1}{2}}$
$\rho < 0, \quad \varepsilon = .05$	$y > -1.876/n^{\frac{1}{2}}$

of each test, and hence the efficiency, will of course decrease. Table 11 lists the optimum and suboptimum minimum efficiencies, and it can be seen that the decrease in efficiency is never critical. It should be noticed that as b increases, the efficiencies of Tests A and D, which are concerned with μ and which use coefficients α_1 and β_1, decrease. The efficiencies of Tests B and F, which are

Table 10. Suboptimum test statistics and acceptance regions for k = 4, b = 3

$$p_1 = .0389 \qquad p_2 = .2160$$
$$p_4 = .9611 \qquad p_3 = .7840$$

Conditions	Acceptance regions
	Test A: $y = .097\,(z_1 + z_4) + .403\,(z_2 + z_3)$
$\mu_2 > \mu_1, \quad \varepsilon = .01$	$y < \mu_1 + 2.513\sigma/n^{\frac{1}{2}}$
$\mu_2 > \mu_1, \quad \varepsilon = .05$	$y < \mu_1 + 1.779\sigma/n^{\frac{1}{2}}$
$\mu_2 < \mu_1, \quad \varepsilon = .01$	$y > \mu_1 - 2.513\sigma/n^{\frac{1}{2}}$
$\mu_2 < \mu_1, \quad \varepsilon = .05$	$y > \mu_1 - 1.779\sigma/n^{\frac{1}{2}}$
	Test B: $y = .179\,(z_4 - z_1) + .235\,(z_3 - z_1)$
$\sigma_2 > \sigma_1, \quad \varepsilon = .01$	$y < \sigma_1(1.001 + 1.850/n^{\frac{1}{2}})$
$\sigma_2 > \sigma_1, \quad \varepsilon = .05$	$y < \sigma_1(1.001 + 1.310/n^{\frac{1}{2}})$
$\sigma_2 < \sigma_1, \quad \varepsilon = .01$	$y > \sigma_1(1.001 - 1.850/n^{\frac{1}{2}})$
$\sigma_2 < \sigma_1, \quad \varepsilon = .05$	$y > \sigma_1(1.001 - 1.310/n^{\frac{1}{2}})$
	Test D: $y = .097\,(z_1 - z_1' + z_4 - z_4')$ $\quad + .403\,(z_2 - z_2' + z_3 - z_3')$
$\theta > 0, \quad \varepsilon = .01$	$y > -2.513\sigma\left(\dfrac{n_1 + n_2}{n_1 n_2}\right)^{\frac{1}{2}}$
$\theta > 0, \quad \varepsilon = .05$	$y > -1.779\sigma\left(\dfrac{n_1 + n_2}{n_1 n_2}\right)^{\frac{1}{2}}$
$\theta < 0, \quad \varepsilon = .01$	$y < 2.513\sigma\left(\dfrac{n_1 + n_2}{n_1 n_2}\right)^{\frac{1}{2}}$
$\theta < 0, \quad \varepsilon = .05$	$y < 1.779\sigma\left(\dfrac{n_1 + n_2}{n_1 n_2}\right)^{\frac{1}{2}}$
	Test F: $y = .179\,(z_4 - z_4' - z_1 + z_1')$ $\quad + .235\,(z_3 - z_3' - z_2 + z_2')$
$\rho > 0, \quad \varepsilon = .01$	$y < 2.617/n^{\frac{1}{2}}$
$\rho > 0, \quad \varepsilon = .05$	$y < 1.852/n^{\frac{1}{2}}$
$\rho < 0, \quad \varepsilon = .01$	$y > -2.617/n^{\frac{1}{2}}$
$\rho < 0, \quad \varepsilon = .05$	$y > -1.852/n^{\frac{1}{2}}$

Table 11. Minimum efficiency under optimum and suboptimum conditions, for k = 4, ε = .01, n = 200

Test	Minimum efficiency			
	Opt	b = 1	b = 2	b = 3
A	.916	.902	.871	.850
B	.838	.754	.789	.797
D	.917	.905	.872	.853
F	.850	.759	.794	.814

concerned with σ and use coefficients α_2 and β_2, increase as b increases. This is consistent with the fact that, as b increases, greater weight is given to $\text{Var}(\hat{\sigma})$ than to $\text{Var}(\hat{\mu})$. This suggests that the choice of b should depend upon the relative importance of the tests to be performed.

Table 12 lists the optimum and suboptimum estimators of ρ. The efficiencies of the suboptimum estimators increase with increasing values of b, which is again con-

sistent with the use of coefficients α_2 and β_2. The loss in efficiency in going from optimum to suboptimum conditions does not appear to be excessive.

Tests A, B, D, and F were applied to the two sets of normal samples $\{x_i\}$ and $\{y_i\}$ referred to in Part VIII. We recall that when the optimum four-quantile test statistics were used for the tests, H_0 was accepted in all cases when H_0 was true and rejected in all cases when H_1 was true, except when Test B was applied to the set $\{y_i\}$. When suboptimum test statistics were used, H_0 was accepted in all cases when H_0 was true. For b = 1, H_0 was rejected in all cases when H_1 was true. For b = 2 and 3, when H_1 was true, H_0 was accepted when Test D was applied, and rejected in all other cases.

Suboptimum quantiles were not considered in connection with Test E because of the special nature of the distribution of the test statistic. Numerical methods are required to determine the acceptance regions and the power of this test. So far, these operations have been carried out only for the special case of ε = .01, n = 200.

Table 12. Estimators of ρ under optimum and suboptimum conditions, for k = 4

Conditions	Estimators of ρ	Eff (n = 200)
Optimum	$\hat{\rho} = \dfrac{.116\,[z_4^2 - (z_4')^2 + z_1^2 - (z_1')^2] + .236\,[z_3^2 - (z_3)^2 + z_2^2 - (z_2')^2]}{\dfrac{5.930}{n} + 3.075}$	.799
b = 1	$\hat{\rho} = \dfrac{.258\,[z_4^2 - (z_4')^2 + z_1^2 - (z_1')^2] + .205\,[z_3^2 - (z_3')^2 + z_2^2 - (z_2')^2]}{\dfrac{5.242}{n} + 2.570}$	.700
b = 2	$\hat{\rho} = \dfrac{.196\,[z_4^2 - (z_4')^2 + z_1^2 - (z_1')^2] + .232\,[z_3^2 - (z_2')^2 + z_2^2 - (z_2')^2]}{\dfrac{5.605}{n} + 2.772}$	.750
b = 3	$\hat{\rho} = \dfrac{.179\,[z_4^2 - (z_4')^2 + z_1^2 - (z_1')^2] + .235\,[z_3^2 - (z_3')^2 + z_1^2 - (z_1')^2]}{\dfrac{5.628}{n} + 2.808}$	.765

X. CONCLUSION

The tests and estimators given in this Report and in Ref. 1 already supply, we believe, impressive evidence of the importance and usefulness of quantiles, both in data compression and in statistical procedures in general when the sample size is large. But the uses of quantiles in statistical analysis surely have not been exhausted. For example, one might certainly prefer to relax the condition in Test A that σ is known, or the condition in Test B that μ is known, because in many practical situations they simply are not known. An apparent drawback to relaxing these conditions is the fact that the distribution of a quantile depends upon the parent density function as well as upon p, and both parameters of a normal distribution must be known in order to specify the distribution completely. This difficulty, however, is not insurmountable, and the solution to the problem provides another example of the advantage to be gained in using suboptimum quantiles.

If one wishes to apply Test A and has no notion of the value of σ, why not *estimate* it, using the same quantiles as those used in the test proper? By using pairs of symmetric quantiles in Tests B_2 and B_4, the condition that μ be known is already unnecessary for these tests. For Test B_1, the condition cannot be relaxed. This follows from the fact that an estimate of σ cannot be obtained from one quantile unless μ is known. The efficiency of the tests of Test A will, of course, suffer but, as very large sample sizes are the rule rather than the exception in space experiments ($n \geq 1000$ is not uncommon), the loss in efficiency will be quite small. A preliminary investigation has already demonstrated this fact.

This procedure can also be applied to Test F and to the estimation of ρ with considerable effectiveness. In general, the two populations will not be standard even if the parameters are known, so that the transformations to the u_i and v_i are given by

$$u_i = \frac{(2)^{1/2}}{2}\left(\frac{x_i - \mu_x}{\sigma_x} + \frac{y_i - \mu_y}{\sigma_y}\right)$$

$$v_i = \frac{(2)^{1/2}}{2}\left(-\frac{x_i - \mu_x}{\sigma_x} + \frac{y_i - \mu_y}{\sigma_y}\right) \tag{17}$$

However, if none of the parameters of the two populations is known, first determine $\hat{\mu}_x$, $\hat{\sigma}_x$, $\hat{\mu}_y$, and $\hat{\sigma}_y$ from the quantiles of the $\{x_i\}$ and $\{y_i\}$, and then substitute each estimate into Eq. 17 for the appropriate unknown parameter. The quantiles of the $\{u_i\}$ and $\{v_i\}$ so obtained can then be used in Test F and also for estimating ρ. Thus one sees that combining estimation and prediction is possible with the use of quantiles, thereby widening the range of simple hypotheses that can be considered.

We have considered only simple hypotheses in this paper, not because composite hypotheses cannot be handled using quantiles, but because the power of a test can only be determined when the hypotheses are simple, and we wished to see how well one can do using quantiles as compared to using the entire sample. The efficiency curves also give us valuable information as to how the efficiency varies with changes in the value of the hypothesized parameters. Moreover, as mentioned previously, the testing of composite hypotheses in many cases requires only a simple modification of the tests of the corresponding simple hypotheses. For example, in Test A, specifying $\sigma_2 \neq \sigma_1$ (instead of, say, $\sigma_2 < \sigma_1$) would result in a two-sided acceptance region of the form, in the one-quantile case,

$$k_2 < z < k_1 \tag{18}$$

where k_1 and k_2 are determined such that the probability of Inequality 18 occurring is equal to $1 - \epsilon$. Nevertheless, a variety of tests of composite hypotheses for which optimum results are not so easily obtained still remain to be considered.

From a practical point of view, the extent of the usefulness of sample quantiles is greatly enhanced when the sample size is large. First of all, as noted previously, normality of the quantiles can be assumed for a sufficiently large n, even when the parent population is not normal but merely satisfies several mild conditions. Secondly, the estimators derived on the basis of a normal parent population are relatively insensitive to deviations from normality (the estimates of σ are somewhat more sensitive than the estimate of μ).

Furthermore, the amount of data compression (a good space program definition of "usefulness") increases as n increases. Now, if one considers that any applied statistical analysis, by whatever name it may go, either is a combination of prediction and estimation or else uses

prediction and/or estimation as tools (and it has been demonstrated that quantiles are indeed useful in many areas of prediction and estimation when n is large), one arrives at the inescapable conclusion that so long as a sufficiently large sample size is available, every phase of statistical activity should be investigated—with a view to determining how useful the substitution of sample quantiles for the entire set of samples will prove to be.

APPENDIX A

Let x and y be independent random variables, distributed $N(m_1, \sigma_1)$ and $N(m_2, \sigma_2)$, respectively. The distribution of $u = x/y$ will be derived.

The joint density of x and y is given by

$$g_1(x, y) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{ -\frac{1}{2}\left[\left(\frac{x - m_1}{\sigma_1}\right)^2 + \left(\frac{x - m_2}{\sigma_2}\right)^2 \right] \right\}$$

Putting

$$x = uv \qquad y = v$$

one sees that, since the Jacobian of the transformation is $|v|$, the joint density of u and v is given by

$$g_2(u, v) = \frac{|v|}{2\pi\sigma_1\sigma_2} \exp\left\{ -\frac{1}{2}\left[\left(\frac{uv - m_1}{\sigma_1}\right)^2 + \left(\frac{v - m_2}{\sigma_2}\right)^2 \right] \right\} \quad \begin{array}{l} -\infty < u < \infty \\ -\infty < v < \infty \end{array}$$

(A-1)

The density function of u, say $h(u)$, can now be obtained by integrating out v in Eq. A-1.

Accordingly,

$$h(u) = \frac{1}{\pi\sigma_1\sigma_2} \int_0^\infty v \exp\left\{ -\frac{1}{2}\left[\left(\frac{uv - m_1}{\sigma_1}\right)^2 + \left(\frac{v - m_2}{\sigma_2}\right)^2 \right] \right\} dv$$

which, by completing the square in v, becomes

$$h(u) = \frac{1}{2\pi} \int_0^\infty v \exp\left\{ -\frac{1}{2}\left[\frac{(\sigma_2^2 u^2 + \sigma_1^2)\left(v - \frac{\sigma_2^2 m_1 u + \sigma_1^2 m_2}{\sigma_2^2 u^2 + u_1^2} \right)^2}{\sigma_1^2\sigma_2^2} + \frac{(m_2 u - m_1)^2}{\sigma_2^2 u^2 + \sigma_1^2} \right] \right\} dv \qquad \text{(A-2)}$$

By use of the transformation

$$v = \sigma_1\sigma_2 w + \frac{\sigma_2^2 m_1 u + \sigma_1^2 m_2}{\sigma_2^2 u^2 + \sigma_1^2}$$

$$dv = \sigma_1\sigma_2 dw$$

Eq. A-2 becomes

$$h(u) = \frac{\sigma_1\sigma_2 \exp\left\{-\frac{1}{2}\left[\frac{(m_2 u - m_1)^2}{\sigma_2^2 u^2 + \sigma_1^2}\right]\right\}}{\pi} \times \int_{\frac{-\sigma_2{}^2 m_1 u - \sigma_1{}^2 m_2}{\sigma_1 \sigma_2 (\sigma_2{}^2 u^2 + \sigma_1{}^2)}}^{\infty} w \exp\left[-\frac{1}{2}(\sigma_2^2 u^2 + \sigma_1^2)w^2\right] dw$$

$$+ \frac{\frac{\sigma_2^2 m_1 u + \sigma_1^2 m_2}{\sigma_2^2 u^2 + \sigma_1^2} \exp\left\{-\frac{1}{2}\left[\frac{(m_2 u - m_1)^2}{\sigma_2^2 u^2 + \sigma_1^2}\right]\right\}}{\pi} \times \int_{\frac{-\sigma_2{}^2 m_1 u - \sigma_1{}^2 m_2}{\sigma_1 \sigma_2 (\sigma_2{}^2 u^2 + \sigma_1{}^2)}}^{\infty} \exp\left[-\frac{1}{2}(\sigma_2^2 u^2 + \sigma_1^2)w^2\right] dw$$

$$= \frac{\sigma_1\sigma_2 \exp\left\{-\frac{1}{2}\left[\frac{(m_2 u - m_1)^2}{\sigma_2^2 u^2 + \sigma_1^2}\right]\right\}}{\pi(\sigma_2^2 u^2 + \sigma_1^2)} \exp\left\{-\frac{1}{2}\left[\frac{(\sigma_2^2 m_1 u + \sigma_1^2 m_2)^2}{\sigma_1^2 \sigma_2^2 (\sigma_2^2 u^2 + \sigma_1^2)}\right]\right\}$$

$$+ \frac{\frac{\sigma_2^2 m_1 u + \sigma_1^2 m_2}{\sigma_2^2 u^2 + \sigma_1^2} \exp\left\{-\frac{1}{2}\left[\frac{(m_2 u - m_1)^2}{\sigma_2^2 u^2 + \sigma_1^2}\right]\right\}}{(2\pi)^{\frac{1}{2}}(\sigma_2^2 u^2 + \sigma_1^2)^{\frac{1}{2}}} \times \left[-1 + 2F\left(\frac{\sigma_2^2 m_1 u + \sigma_1^2 m_2}{\sigma_1 \sigma_2 (\sigma_2^2 u^2 + \sigma_1)^{\frac{1}{2}}}\right)\right]$$

$$= \frac{\sigma_1\sigma_2 \exp\left[-\frac{1}{2}\left(\frac{\sigma_1^2 m_2^2 + \sigma_2 m_1^2}{\sigma_1^2 \sigma_2^2}\right)\right]}{\pi(\sigma_2^2 u^2 + \sigma_1^2)} + \frac{\sigma_2^2 m_1 u + \sigma_1^2 m_2}{(2\pi)^{\frac{1}{2}}(\sigma_2^2 u^2 + \sigma_1^2)^{\frac{3}{2}}}$$

$$\times \exp\left\{-\frac{1}{2}\left[\frac{(m_2 u - m_1)^2}{\sigma_2^2 u^2 + \sigma_1^2}\right]\right\} \times \left[-1 + 2F\left(\frac{\sigma_2^2 m_1 u + \sigma_1^2 m_2}{\sigma_1 \sigma_2 (\sigma_2^2 u^2 + \sigma_1^2)^{\frac{1}{2}}}\right)\right] \quad (-\infty < u < \infty)$$

If $m_1 = m_2 = 0$, it is readily seen that the density function of $u = x/y$ has the relatively simple form

$$h(u) = \frac{\sigma_1\sigma_2}{\pi(\sigma_2^2 u^2 + \sigma_1^2)} \quad (-\infty < u < \infty)$$

APPENDIX B

Let x and y be random variables distributed $N(m_1, \sigma_1)$ and $N(m_2, \sigma_2)$, respectively, and let ρ denote the correlation between x and y. The expected value of the product $x^2 y^2$, ($E(x^2 y^2)$) will be derived by two different methods when $\rho \neq 0$. If $\rho = 0$, it is easily seen that

$$E(x^2 y^2) = E(x^2) E(y^2) = \sigma_1^2 \sigma_2^2 + \sigma_1^2 m_2^2 + \sigma_2^2 m_1^2 + m_1^2 m_2^2$$

If $\rho \neq 0$, define

$$u = \frac{\sigma_2}{\sigma_1} x + y$$

$$v = -\frac{\sigma_2}{\sigma_1} x + y$$

Then one has

$$E(u) = \frac{\sigma_2}{\sigma_1} m_1 + m_2 \qquad E(v) = -\frac{\sigma_2}{\sigma_1} m_1 + m_2$$

$$\text{Var}(u) = 2\sigma_2^2 (1 + \rho) \qquad \text{Var}(v) = 2\sigma_2^2 (1 - \rho)$$

$$\text{Cov}(u, v) = 0$$

Thus u and v are normally distributed and independent, and x and y can be expressed in terms of u and v by the inverse transformation

$$x = \frac{\sigma_1}{2\sigma_2} (u - v)$$

$$y = \frac{\sigma_1}{2\sigma_2} (u + v)$$

Then one has

$$E(x^2 y^2) = \frac{\sigma_1^2}{16\sigma_2^2} E[u^4 - 2u^2 v^2 + v^4] = \frac{\sigma_1^2}{16\sigma_2^2} \left\{ 12\sigma_2^4 (1 + \rho)^2 + \left(\frac{\sigma_2}{\sigma_1} m_1 + m_2 \right)^4 + 12\sigma_2^2 (1 + \rho) \left(\frac{\sigma_2}{\sigma_1} m_1 + m_2 \right)^2 \right.$$

$$-2 \left[2\sigma_2^2 (1 + \rho) + \left(\frac{\sigma_2}{\sigma_1} m_1 + m_2 \right)^2 \right] \left[2\sigma_2^2 (1 - \rho) + \left(-\frac{\sigma_2}{\sigma_1} m_1 + m_2 \right)^2 \right]$$

$$\left. + 12\sigma_2^4 (1 - \rho)^2 + \left(-\frac{\sigma_2}{\sigma_1} m_1 + m_2 \right)^4 + 12\sigma_2^2 (1 - \rho) \left(-\frac{\sigma_2}{\sigma_1} m_1 + m_2 \right) \right\}$$

$$= \sigma_1^2 \sigma_2^2 (1 + 2\rho^2) + \sigma_1^2 m_2^2 + \sigma_2^2 m_1^2 + m_1^2 m_2^2 + 4 m_1 m_2 \sigma_1 \sigma_2 \rho$$

The second method uses the characteristic function of the joint distribution of x and y, $\phi(t_1, t_2)$, which is defined by

$$\phi(t_1, t_2) = E(e^{i t_1 x + i t_2 y}) = \int\int\limits_{-\infty}^{\infty} e^{i t_1 x + i t_2 y} g(x, y) \, dx \, dy \qquad \text{(A-3)}$$

From Ref. 4, p. 288, we find that the evaluation of the integral results in

$$\phi\left(t_1, t_2\right) = \exp\left[i\left(m_1 t_1 + m_2 t_2\right) - \frac{1}{2}\left(\sigma_1^2 t_1^2 + 2\sigma_1\sigma_2\rho t_1 t_2 + \sigma_2^2 t_2^2\right)\right]$$

By taking successive partial derivatives of $\phi\left(t_1, t_2\right)$ as defined in Eq. A-3 and evaluating the final result at $t_1 = t_2 = 0$, it is readily seen that, in general,

$$E\left(x^r y^s\right) = \left(-i\right)^{r+s}\left.\frac{\partial^{r+s}\phi}{\partial t^r \partial t^s}\right|_{\substack{t_1 = 0 \\ t_2 = 0}}$$

Thus for the particular case $r = s = 2$, one obtains

$$\frac{\partial\phi}{\partial t_1} = \left(im_1 - \sigma_1^2 t_1 - \sigma_1\sigma_2\rho t_2\right)\phi\left(t_1, t_2\right)$$

$$\frac{\partial^2\phi}{\partial t_1^2} = \left[\left(im_1 - \sigma_1^2 t_1 - \sigma_1\sigma_2\rho t_2\right)^2 - \sigma_1^2\right]\phi\left(t_1, t_2\right)$$

$$\frac{\partial^3\phi}{dt_1^2 \partial t_2} = \left\{-2\sigma_1\sigma_2\rho\left(im_1 - \sigma_1^2 t_1 - \sigma_1\sigma_2\rho t_2\right) + \left(im_2 - \sigma_2^2 t_2 - \sigma_1\sigma_2 t_1\right)\left[\left(im_1 - \sigma_1^2 t_1 - \sigma_1\sigma_2\rho t_2\right)^2 - \sigma_1^2\right]\right\}\phi\left(t_1, t_2\right)$$

$$\frac{\partial^4\phi}{\partial t_1^2 \partial t_2^2} = \left\{2\sigma_1^2\sigma_2^2\rho^2 - \sigma_2^2\left[\left(im_1 - \sigma_1^2 t_1 - \sigma_1\sigma_2\rho t_2\right)^2 - \sigma_1^2\right]\right.$$

$$\left. - 2\sigma_1\sigma_2\rho\left(im_2 - \sigma_2^2 t_2 - \sigma_1\sigma_2 t_1\right)\left(im_1 - \sigma_1^2 t_1 - \sigma_1\sigma_2\rho t_2\right)\phi\left(t_1, t_2\right) + \left(im_2 - \sigma_2^2 t_2 - \sigma_1\sigma_2\rho t_2\right)\frac{\partial^3\phi}{\partial t_1^2 \partial t_2}\right\}$$

and, finally

$$E\left(x^2 y^2\right) = \left.\frac{\partial^4\phi}{\partial t_1^2 t_2^2}\right|_{\substack{t_1 = 0 \\ t_2 = 0}} = 2\sigma_1^2\sigma_2^2\rho^2 - \sigma_2^2\left(im_1\right)^2 + \sigma_1^2\sigma_2^2 - 2\sigma_1\sigma_2\rho\left(i^2 m_1 m_2\right) + im_2\left[-2\sigma_1\sigma_2\rho im_1 + i^3 m_1^2 m_2 - im_2\sigma_1^2\right]$$

$$= \sigma_1^2\sigma_2^2\left(1 + 2\rho^2\right) + \sigma_1^2 m_2^2 + \sigma_2^2 m_1^2 + m_1^2 m_2^2 + 4m_1 m_2 \sigma_1\sigma_2\rho$$

It is interesting to note that when $m_1 = m_2 = 0$, one has

$$\frac{E\left(x^2 y^2\right) - E\left(x^2\right)E\left(y^2\right)}{\left(\text{Var}\left[x^2\right]\text{Var}\left[y^2\right]\right)^{1/2}} = \frac{\sigma_1^2\sigma_2^2\left(1 + 2\rho^2\right) - \sigma_1^2\sigma_2^2}{2\sigma_1^2\sigma_2^2} = \rho^2$$

so that, for this special case, the correlation between x^2 and y^2 is the square of the correlation between x and y.

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